Working with a difference quotient involving a square root

Suppose $f(x) = \sqrt{x}$ and suppose we want to simplify the differnce quotient

$$\frac{f(x+h) - f(x)}{h}$$

as much as possible (say, to eliminate the h in the denominator).

Substituting the definition of f into the quotient, we have

$$\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

at which point we are stuck, as far as basic algebraic manipulations go.

To the rescue, however, comes *the conjugate*.

For any expression of the form $\sqrt{A} - \sqrt{B}$, we say its conjugate is $\sqrt{A} + \sqrt{B}$, and vice versa: the conjugate of the latter is the former: we get to the expressions conjugate by simply changing the sign of the operation between the two square root expressions (plus to minus, or minus to plus).

By writing the number 1 as the expression's conjugate divided by itself, we get a powerful tool for manipulating these types of expressions.

With

$$\frac{\sqrt{x+h} - \sqrt{x}}{h},$$

the conjugate we want to use is $\sqrt{x+h} + \sqrt{x}$, so we multiply our expression by the conjugate over itself:

$$\frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}.$$

The key idea is that the numerators multiply in a nice way.

Note that the two numerators together have the form

$$(A - B) \cdot (A + B)$$

which is equal to $A^2 - B^2$ (you might recall the phrase *difference of squares*). The squaring eliminates the square roots from the numerator.

As a result, our expression above becomes

$$\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \frac{h}{h} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x+h} + \sqrt{x}}.$$

Thus, we have shown that

$$\frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{1}{\sqrt{x+h} + \sqrt{x}}.$$

This is as simplified as we can make it, and it has the advantage over the original expression in that it has no h multiplier in the denominator (which will be a consideration when you see this sort of thing again in Calculus).