Composition Example

Let $f(x) = |x-1|$. So

$$f(x) = |x-1| = \begin{cases} x-1 & \text{if } x-1 \geq 0, \\ -(x-1) & \text{if } x-1 < 0 \end{cases} = \begin{cases} x-1 & \text{if } x \geq 1, \\ -x+1 & \text{if } x < 1. \end{cases}$$

Suppose we are interested in the composition of f with itself, i.e., the function $f(f(x))$. Now, we have

$$f(f(x)) = f(|x-1|) = \begin{cases} |x-1|-1 & \text{if } |x-1| \geq 1, \\ -x+1 & \text{if } |x-1| < 1 \end{cases}$$

The inequalities above are not the most convenient (for instance, if we wanted to graph $f(f(x))$. So we simplify:

$$|x-1| \geq 1$$

if, and only if, $x \geq 2$ or $x \leq 0$ (why?). Similarly,

$$|x-1| < 1$$

is equivalent to $0 < x < 2$. So we can rewrite $f(f(x))$:

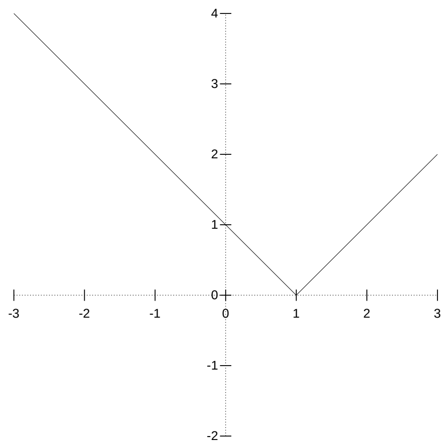
$$f(f(x)) = \begin{cases} |x-1|-1 & \text{if } x \geq 2, \\ -|x-1|+1 & \text{if } 0 < x < 2, \\ |x-1|-1 & \text{if } x \leq 0 \end{cases}$$

$$= \begin{cases} x-1-1 & \text{if } x \geq 2 \text{ (since } |x-1| > 0 \text{ if } x \geq 2), \\ -|x-1|+1 & \text{if } 0 < x < 2, \\ -(x-1)-1 & \text{if } x \leq 0 \text{ (since } |x-1| < 0 \text{ if } x \leq 0 \end{cases}$$

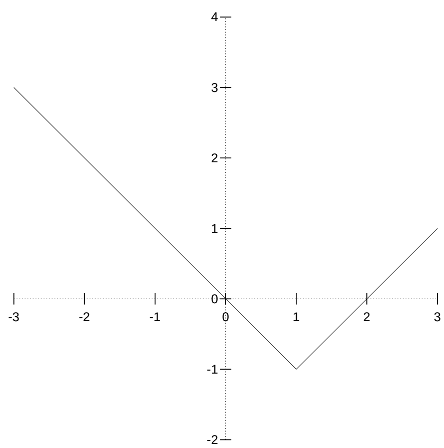
$$= \begin{cases} x-2 & \text{if } x \geq 2, \\ -(x-1)+1 & \text{if } 1 \leq x < 2, \\ (x-1)+1 & \text{if } 0 < x < 1, \\ -(x-1)-1 & \text{if } x \leq 0 \end{cases}$$

$$= \begin{cases} x-2 & \text{if } x \geq 2, \\ -x+2 & \text{if } 1 \leq x < 2, \\ x & \text{if } 0 < x < 1, \\ -x & \text{if } x \leq 0 \end{cases}$$

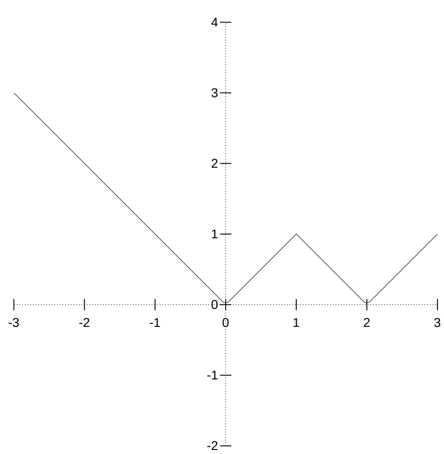
Another way to look at this is graphically. First, we have a graph of $f(x) = |x-1|$:



Then, let $g(x) = |x - 1| - 1 = f(x) - 1$. This looks just like $f(x)$ shifted down one unit:



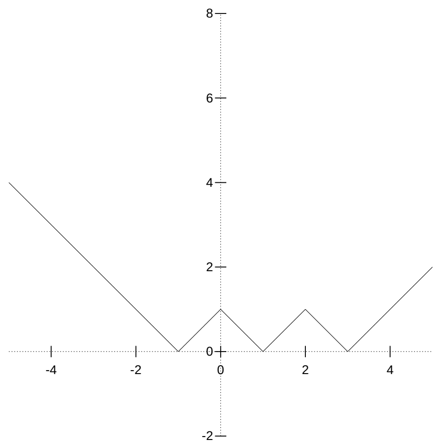
Then let $h(x) = ||x - 1| - 1|$. This looks like $g(x)$, except that where it was negative it has now been flipped positive, across the x-axis:



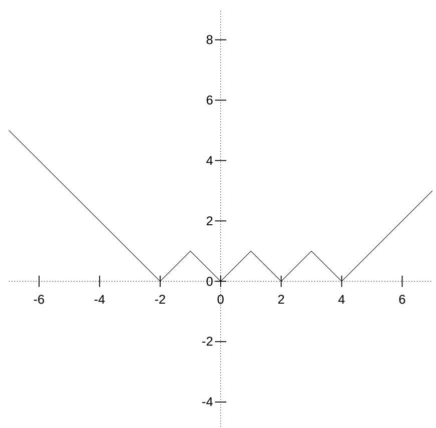
This is $f(f(x)))$.

If we continue composing f with itself, a pattern emerges.

The graph below is $f(f(f(x)))$:



This is the graph of $f(f(f(f(x))))$:



Each step adds another "tooth".