

# A Table of Isoperimetric Ratios

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For a simple closed plane curve, the **isoperimetric ratio** is

$$\frac{P^2}{A}$$

where  $P$  is the length of the curve, and  $A$  is the area enclosed by the curve. This ratio is a dimensionless constant that is invariant under scaling: changing the *size* of the shape does not change this ratio.

The table below gives this ratio for a variety of shapes.

One can prove that the circle is the plane curve with the smallest (and hence, smallest possible) isoperimetric ratio, and so the table begins with the circle.

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I started this table as a result of a homework problem in our precalculus class at the University of Washington. The problem asks the student to consider a situation in which we have a piece of wire of known length; we want to cut the wire and bend one resulting piece into a circle and the other piece into a square. The question is: where should we cut the wire so that the area of the circle plus the area of the square is minimized?

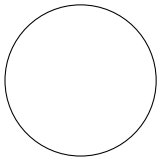
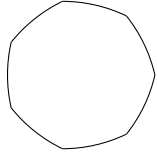
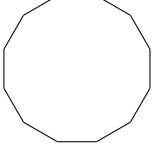
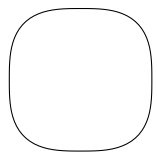
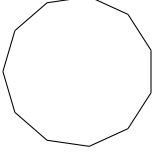
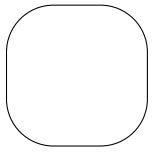
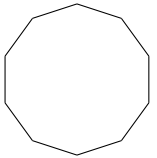
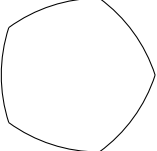
When writing exam questions of a similar flavor, the relative “efficiencies” of different planar shapes come under consideration.

For any wire length, if the two shapes have isoperimetric ratios  $r_1$  and  $r_2$ , then the wire should be cut into two pieces with lengths in the ratio  $r_1 : r_2$  if we wish to minimize the total area.

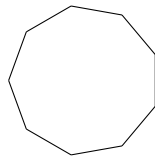
## Cited by

This table is cited in the following paper:

Hirvonen, Petri & Boissonière, Gabriel & Fan, Zheyong & Achim, Cristian & Provas, Nikolas & Elder, Ken & Ala-Nissila, Tapio. (2018). Grain extraction and microstructural analysis method for two-dimensional poly and quasicrystalline solids. *Physical Review Materials*. 2. 10.1103/PhysRevMaterials.2.103603.

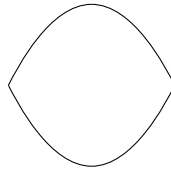
circle		$4\pi \approx 12.56637061$
Reuleaux heptagon		$\approx 12.78538634$
regular 12-gon		$96 - 48\sqrt{3} \approx 12.86156124$
super ellipse $ x ^3 +  y ^3 = 1$		$\approx 12.87612775$
regular 11-gon		$44 \tan \frac{\pi}{11} \approx 12.91956568$
square with quarter circle rounded corners (radius of corners is one-third of overall width)		$\frac{(4 + 2\pi)^2}{5 + \pi} \approx 12.98810988$
regular 10-gon		$40 \tan \frac{\pi}{10} \approx 12.99678784$
Reuleaux pentagon		$\approx 13.01205315$

regular 9-gon



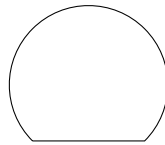
$$36 \tan \frac{\pi}{9} \approx 13.10292843$$

optimum region  
bounded by symmetric  
parabolic arcs (parabolas  
are approximately  
 $y = 0.97300151x^2$   
on the interval  
 $(-0.97300151, 0.97300151)$ )



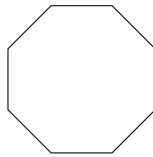
$$\approx 13.11934320$$

three-quarter truncated  
circular arc



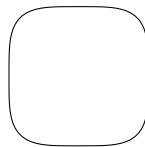
$$\approx 13.14170265$$

regular octagon (equiv-  
alent to square aug-  
mented by four opti-  
mal isosceles triangles)



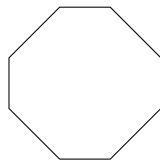
$$32(\sqrt{2} - 1) \approx 13.25483399$$

super ellipse  
 $|x|^4 + |y|^4 = 1$



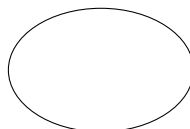
$$\approx 13.28104122$$

square with hacked off  
corners (horizontal and  
vertical edges are one-  
third overall width)

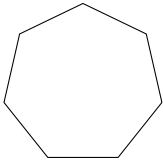
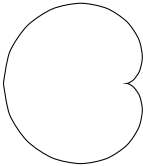
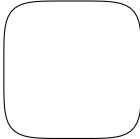
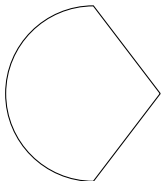
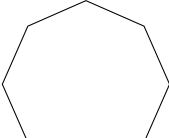
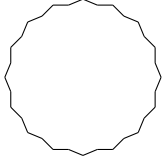
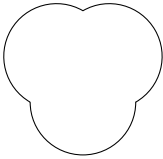
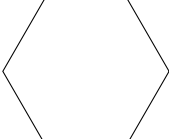


$$\frac{(4 + 4\sqrt{2})^2}{7} \approx 13.32211914$$

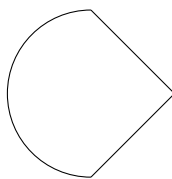
ellipse, 3:2



$$\approx 13.35374554$$

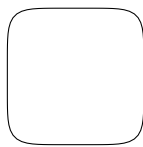
regular septagon		$28 \tan \frac{\pi}{7} \approx 13.48408932$
cardioid		$\frac{128}{3\pi} \approx 13.58122181$
super ellipse $ x ^5 +  y ^5 = 1$		$\approx 13.61890282$
semicircle with optimal adjoined isosceles tri- angle		$\approx 13.71361987$
square with three opti- mal adjoined isosceles triangles		$\approx 13.75752837$
16 pointed star		$\approx 13.77927647$
equilateral triangle with three adjoined semicircles		$\approx 13.78342300$
regular hexagon		$\frac{24}{\sqrt{3}} \approx 13.85640646$

semicircle with adjoined right isosceles triangle



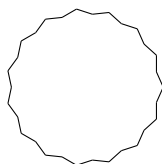
$$\frac{2(2\sqrt{2} + \pi)^2}{\pi + 2} \approx 13.86385058$$

super ellipse  
 $|x|^6 + |y|^6 = 1$



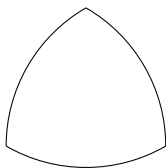
$$\approx 13.88994096$$

15 pointed star



$$\approx 13.96032741$$

Reuleaux triangle



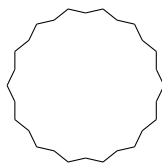
$$\frac{2\pi^2}{\pi - \sqrt{3}} \approx 14.00398921$$

equilateral triangle with two adjoined semicircles



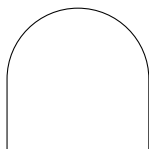
$$\frac{4(1 + \pi)^2}{\pi + \sqrt{3}} \approx 14.07800126$$

14 pointed star



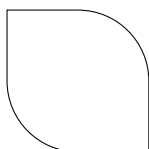
$$\approx 14.18654408$$

semicircle with optimal adjoined rectangle



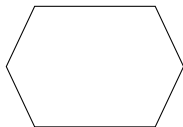
$$8 + 2\pi \approx 14.2831853071$$

square with two equally, optimally rounded corners (radii are half the side length)



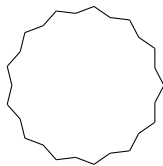
$$8 + 2\pi \approx 14.2831853071$$

square with two optimal adjoined isosceles triangles



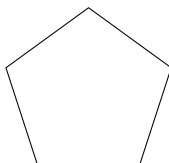
$$\approx 14.35072219$$

13 pointed star



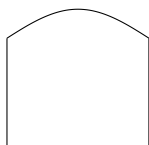
$$\approx 14.47485646$$

regular pentagon



$$\frac{100}{\sqrt{25 + 10\sqrt{5}}} \approx 14.53085056$$

optimal region bounded by  $\frac{1}{x^2 + 1}$ , the  $x$ -axis, and  $x = \pm a$ , where  $a \approx 0.5181167$



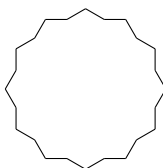
$$\approx 14.695335322$$

square with two adjoined semicircles



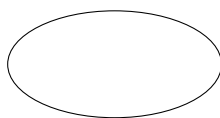
$$\frac{4(2 + \pi)^2}{4 + \pi} \approx 14.80676722$$

12 pointed star



$$\approx 14.85125168$$

ellipse, 2:1



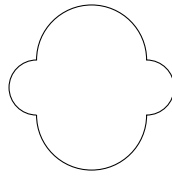
$$\approx 14.93924249$$

square with three adjoined semicircles



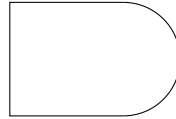
$$\frac{2(2 + 3\pi)^2}{8 + 3\pi} \approx 14.98160283$$

rectangle, 2:1, with four  
adjoined semicircles



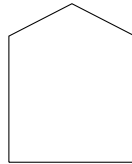
$$\frac{36\pi^2}{8 + 5\pi} \approx 14.98676855$$

square with adjoined  
semicircle



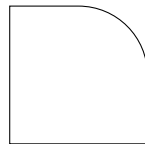
$$\frac{2(6 + \pi)^2}{8 + \pi} \approx 15.00121550$$

square with optimal  
adjoined isosceles  
triangle



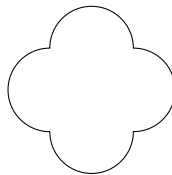
$$\approx 15.07344594$$

square with single op-  
timally rounded corner  
(radius is one half the  
side length)



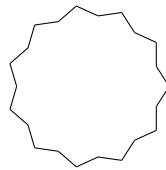
$$12 + \pi \approx 15.14159265$$

square with four ad-  
joined semicircles



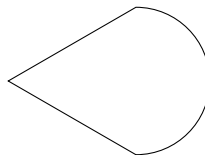
$$\frac{8\pi^2}{2 + \pi} \approx 15.35649370$$

11 pointed star



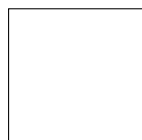
$$\approx 15.35751730$$

equilateral triangle  
with adjoined semicir-  
cle



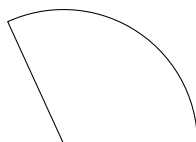
$$\frac{2(4 + \pi)^2}{\pi + 2\sqrt{3}} \approx 15.44193344$$

square



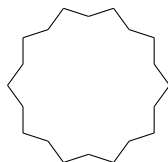


circular sector, two radians (optimal circular sector)



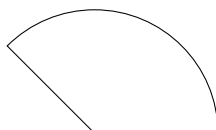
16

10 pointed star



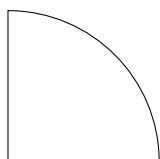
$\approx 16.06491327$

circular sector,  $3\pi/4$  radians



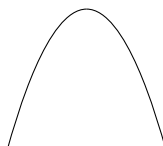
$\frac{(8 + 3\pi)^2}{6\pi} \approx 16.10769443$

quarter circular sector



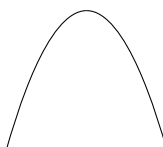
$\frac{4(2 + \frac{\pi}{2})^2}{\pi} \approx 16.23455083$

optimal parabolic chunk (region bounded by  $y = x^2$  and  $y \approx 3.127903$ )



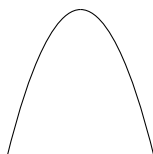
$\approx 16.48263518$

parabolic chunk: region bounded by  $y = x^2$  and  $y = 3$



$\approx 16.48509413$

parabolic chunk: region bounded by  $y = x^2$  and  $y = 4$



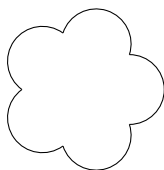
$\approx 16.56740039$

rectangle, 3:2 sides



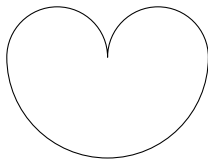
$\frac{50}{3} = 16.\bar{6}$

regular pentagon with  
adjoined semicircles



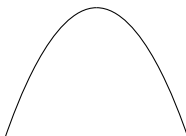
$$\frac{50\pi^2}{5\pi + 2\sqrt{25 + 10\sqrt{5}}} \approx 16.74415928$$

semicircle augmented  
by two semicircles, I



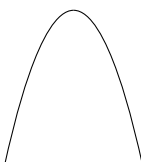
$$\frac{16}{3}\pi \approx 16.75516081$$

parabolic chunk: re-  
gion bounded by  $y =$   
 $x^2$  and  $y = 2$



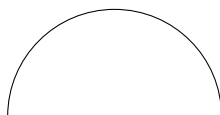
$$\approx 16.76938845$$

parabolic chunk: re-  
gion bounded by  $y =$   
 $x^2$  and  $y = 5$



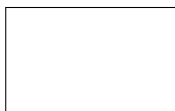
$$\approx 16.79033103$$

semicircle



$$\frac{2(2 + \pi)^2}{\pi} \approx 16.82966439$$

golden rectangle (rect-  
angle with side ratio  
 $1:\phi$  where  $\phi = \frac{1}{2}(1 +$   
 $\sqrt{5})$



$$8\phi + 4 \approx 16.944271909$$

rectangle with side ra-  
tio  $1:1.63712909\dots$



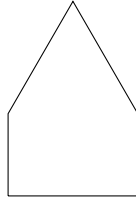
$$\approx 16.99181787$$

rectangle with side ratio 1:1.63712909... with adjoined semicircles



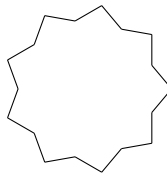
$$\approx 16.99181787$$

equilateral triangle with adjoined optimal rectangle



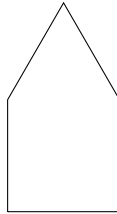
$$4(6 - \sqrt{3}) \approx 17.07179676$$

9 pointed star



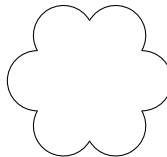
$$\approx 17.10465828$$

square with adjoined equilateral triangle



$$\frac{100}{4 + \sqrt{3}} \approx 17.44576302$$

regular hexagon with adjoined semicircles



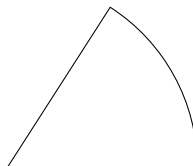
$$\frac{12\pi^2}{\pi + 2\sqrt{3}} \approx 17.92926648$$

rectangle, 2:1



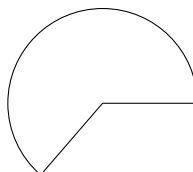
$$18$$

circular sector, one radian



$$18$$

circular sector, four radians



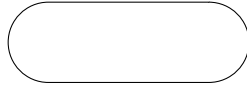
$$18$$

rectangle, 2:1, with one  
adjoined semicircle



$$\frac{2(10 + \pi)^2}{16 + \pi} \approx 18.044627800654$$

rectangle, 2:1, with two  
adjoined semicircles



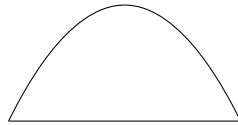
$$\frac{4(4 + \pi)^2}{8 + \pi} \approx 18.31061221$$

parabolic chunk: re-  
gion bounded by  $y =$   
 $x^2$  and  $y = 10$



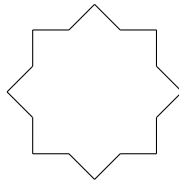
$$\approx 18.38876821$$

parabolic chunk: re-  
gion bounded by  $y =$   
 $x^2$  and  $y = 1$



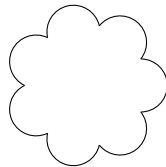
$$\approx 18.43547307$$

8 pointed star



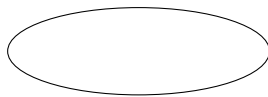
$$64 - 4^{11/4} \approx 18.74516600$$

regular septagon with  
adjoined semicircles



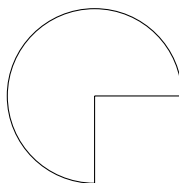
$$\approx 18.94192830$$

ellipse, 3:1



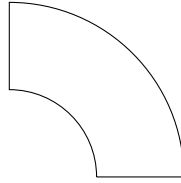
$$\approx 18.95220996$$

three-quarter circular  
sector



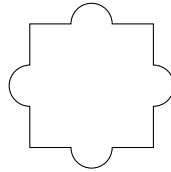
$$\frac{(4 + 3\pi)^2}{3\pi} \approx 19.12243068$$

quarter circular sector  
with half-radius quar-  
ter circular sector re-  
moved



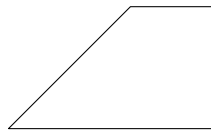
$$\frac{(4 + 3\pi)^2}{3\pi} \approx 19.12243068$$

square with an ad-  
joined semicircle on  
each side's middle  
third



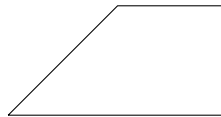
$$\frac{8(\pi + 4)^2}{18 + \pi} \approx 19.29933907$$

isosceles right triangle  
with optimal adjoined  
rectangle ( $1 : \frac{\sqrt{2}}{2}$ )



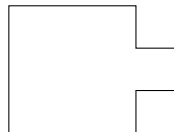
$$8 + 8\sqrt{2} \approx 19.31370849$$

isosceles right triangle  
with adjoined square



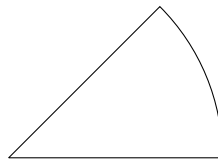
$$\frac{2(4 + \sqrt{2})^2}{3} \approx 19.54247233$$

square with square ad-  
joined to middle-third  
of one side



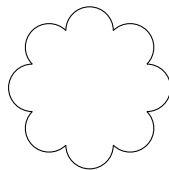
$$\frac{98}{5} = 19.6$$

circular sector,  $\pi/4$  ra-  
dians



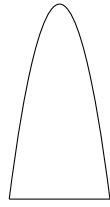
$$\frac{(8 + \pi)^2}{2\pi} \approx 19.756712684$$

regular octagon with  
adjoined semicircles



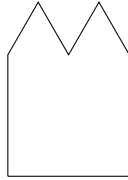
$$\approx 19.81346029$$

parabolic chunk: region bounded by  $y = x^2$  and  $y = 15$



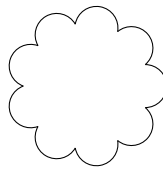
$$\approx 20.00749094$$

square with two equilateral triangles adjoined to one side



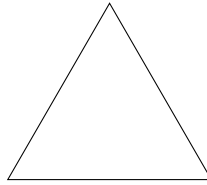
$$\frac{200}{8 + \sqrt{3}} \approx 20.55065309$$

regular nonagon with adjoined semicircles



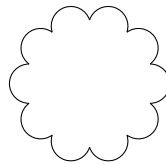
$$\approx 20.569895$$

equilateral triangle



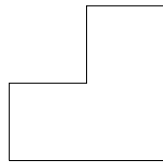
$$\frac{36}{\sqrt{3}} \approx 20.78460969$$

regular decagon with adjoined semicircles



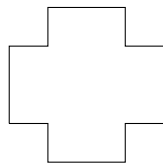
$$\approx 21.231897$$

two step unit staircase



$$\frac{64}{3} = 21.\bar{3}$$

square with quarter-sized squares removed from each corner (aka order-2 Aztec diamond)



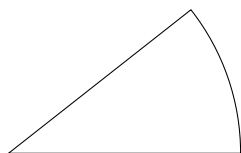
$$\frac{64}{3} = 21.\bar{3}$$

rectangle, 3:1



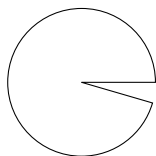
$$\frac{64}{3} = 21.\bar{3}$$

circular sector,  $2/3$  radians



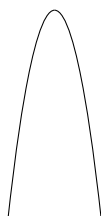
$$\frac{64}{3} = 21.\bar{3}$$

circular sector, 6 radians



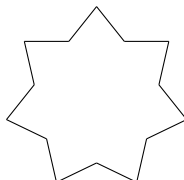
$$\frac{64}{3} = 21.\bar{3}$$

parabolic chunk: region bounded by  $y = x^2$  and  $y = 20$



$$\approx 21.50079173$$

7 pointed star



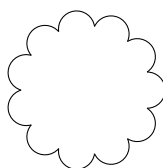
$$\approx 21.62680012$$

rectangle, 3:1, with one adjoined semicircle



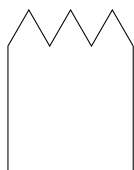
$$\frac{2(14 + \pi)^2}{24 + \pi} \approx 21.65194964$$

regular 11-gon with adjoined semicircles



$$\approx 21.81573534$$

square with three equilateral triangles adjoined to one side



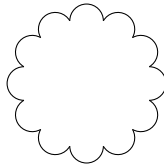
$$\frac{300}{12 + \sqrt{3}} \approx 21.84670041$$

rectangle, 3:1, with two adjoined semicircles



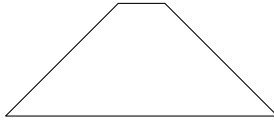
$$\frac{4(6 + \pi)^2}{12 + \pi} \approx 22.076598718$$

regular 12-gon with adjoined semicircles



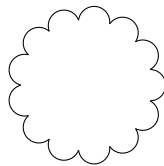
$$\approx 22.33427625$$

optimal "45 degree" trapezoid (base= 1, height=  $\frac{1}{\sqrt{2}+1}$ ) (aka optimal truncated isosceles right triangle aka optimal rectangle with two adjoined isosceles right triangles)



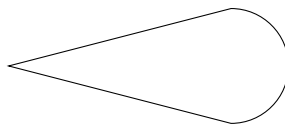
$$16\sqrt{2} \approx 22.62741699796$$

regular 13-gon with adjoined semicircles



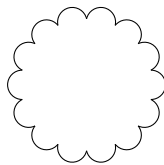
$$\approx 22.79777267$$

isosceles triangle, 2:1 sides, with adjoined semicircle



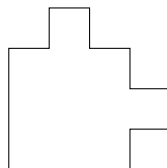
$$\frac{2(8 + \pi)^2}{\pi + 2\sqrt{15}} \approx 22.80310635$$

regular 14-gon with adjoined semicircles



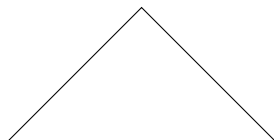
$$\approx 23.21447316$$

square with squares adjoined to middle third of two sides



$$\frac{256}{11} = 23.\overline{27}$$

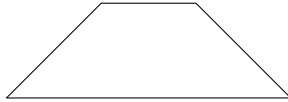
isosceles right triangle



$$2(2 + \sqrt{2})^2 \approx 23.31370849$$

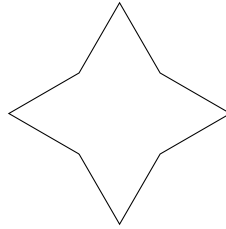


square with two ad-joined isosceles right triangles



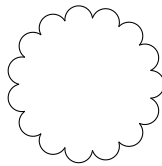
$$2(2 + \sqrt{2})^2 \approx 23.31370849$$

square with four ad-joined equilateral triangles



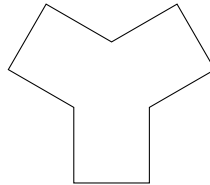
$$\frac{64}{1 + \sqrt{3}} \approx 23.42562584$$

regular 15-gon with ad-joined semicircles



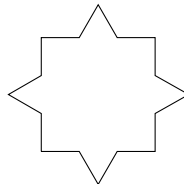
$$\approx 23.59107664$$

equilateral triangle with three ad-joined squares



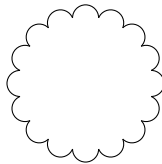
$$\frac{324}{12 + \sqrt{3}} \approx 23.59443644$$

square with an ad-joined equilateral triangles on each side's middle third



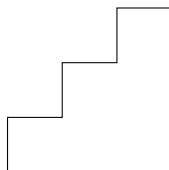
$$\frac{256}{9 + \sqrt{3}} \approx 23.85378196$$

regular 16-gon with ad-joined semicircles



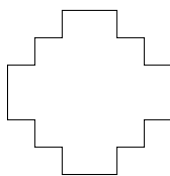
$$\approx 23.93307364$$

three-step unit staircase



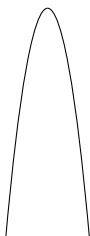
24

order-3 Aztec diamond



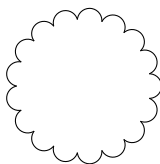
24

parabolic chunk: region bounded by  $y = x^2$  and  $y = 30$



$\approx 24.15608124$

regular 17-gon with adjoined semicircles



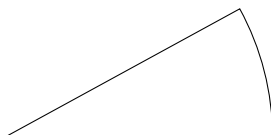
$\approx 24.2450034$

rectangle, 4:1



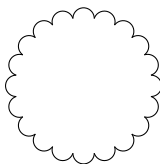
25

circular sector, 1/2 radian



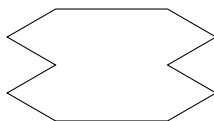
25

regular 20-gon with adjoined semicircles



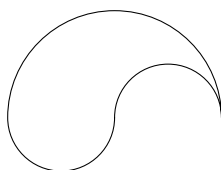
$\approx 25.03530855$

square with two equilateral triangles adjoined to two sides



$\frac{144}{4 + \sqrt{3}} \approx 25.1218987469293$

semicircle augmented with two semicircles, II



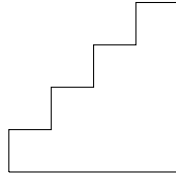
$8\pi \approx 25.13274122$

rectangle, 4:1, with one  
adjoined semicircle



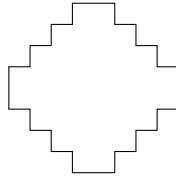
$$\frac{2(18 + \pi)^2}{32 + \pi} \approx 25.43805821$$

four-step unit staircase



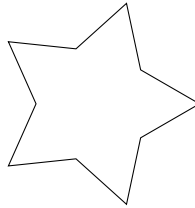
$$\frac{128}{5} = 25.6$$

order-4 Aztec diamond



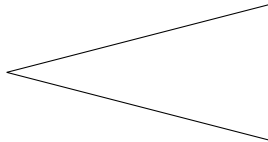
$$\frac{128}{5} = 25.6$$

regular pentagon with  
adjoined equilateral tri-  
angles



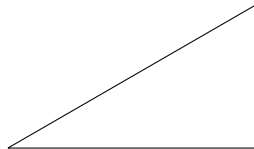
$$\approx 25.73644244$$

isosceles triangle, 2:1  
sides



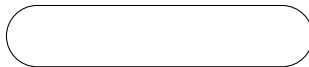
$$\frac{100}{\sqrt{15}} \approx 25.81988897$$

right triangle, 30-60-90



$$12 + 8\sqrt{3} \approx 25.85640646$$

rectangle, 4:1, with two  
adjoined semicircles



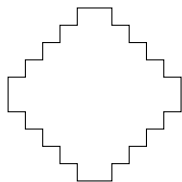
$$\frac{4(8 + \pi)^2}{16 + \pi} \approx 25.94038836893$$

parabolic chunk: re-  
gion bounded by  $y =$   
 $x^2$  and  $y = 40$



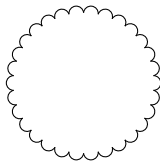
$$\approx 26.48326571$$

order-5 Aztec diamond



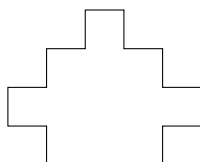
$$\frac{80}{3} = 26.\bar{6}$$

regular 30-gon with adjoined semicircles



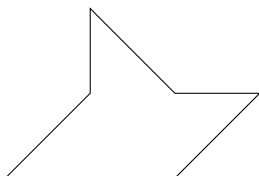
$$\approx 26.71031462$$

square with squares adjoined to the middle thirds of three sides



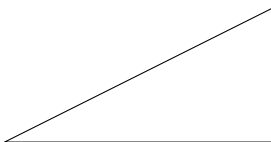
$$27$$

square with three adjoined right isosceles triangles



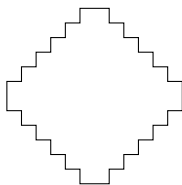
$$\frac{68 + 48\sqrt{2}}{5} \approx 27.17645019$$

right triangle, 2:1 legs



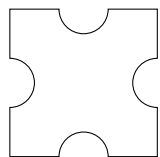
$$(3 + \sqrt{5})^2 = 14 + 6\sqrt{5} \approx 27.41640786$$

order-6 Aztec diamond



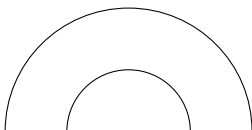
$$\frac{192}{7} = 27.\overline{428571}$$

square with one-third diameter semicircles removed from each side



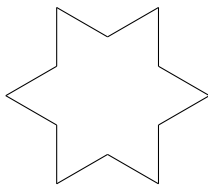
$$\frac{2(8 + 2\pi)^2}{18 - \pi} \approx 27.46046434$$

semicircle with half-diameter semicircle removed



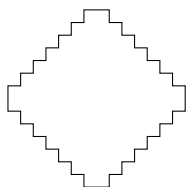
$$\frac{2(3\pi + 2)^2}{3\pi} \approx 27.69838228$$

6 pointed star (Koch curve, stage 2; regular hexagon with adjoined equilateral triangles)



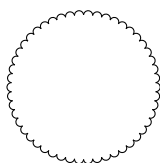
$$\sqrt{768} \approx 27.71281292$$

order-7 Aztec diamond



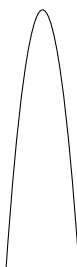
$$28$$

regular 50-gon with adjoined semicircles



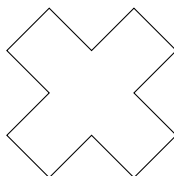
$$\approx 28.25482853$$

parabolic chunk: region bounded by  $y = x^2$  and  $y = 50$



$$\approx 28.57579596$$

5-square cross



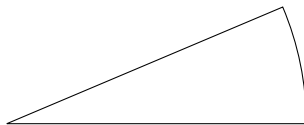
$$\frac{144}{5} = 28.8$$

rectangle, 5:1



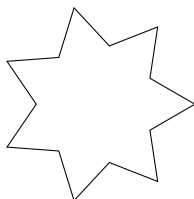
$$\frac{144}{5} = 28.8$$

circular sector,  $2/5$  radian



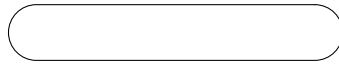
$$\frac{144}{5} = 28.8$$

regular septagon with adjoined equilateral triangles



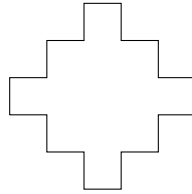
$$\approx 29.40734584$$

rectangle, 5:1, with two  
adjoined semicircles



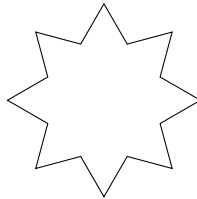
$$\frac{4(10 + \pi)^2}{20 + \pi} \approx 29.851265651086$$

13-square diamond



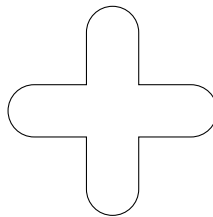
$$\frac{400}{13} \approx 30.76923076$$

regular octagon with  
adjoined equilateral tri-  
angles



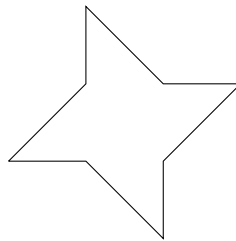
$$\approx 30.87116222$$

5-square cross with  
four adjoined semicir-  
cles



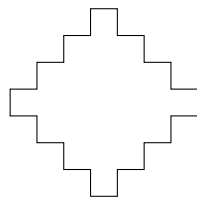
$$\frac{8(4 + \pi)^2}{\pi + 10} \approx 31.04789318$$

square with four ad-  
joined isosceles right  
triangles



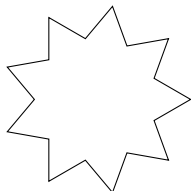
$$16 + \frac{32}{3}\sqrt{2} \approx 31.08494466$$

25-square diamond



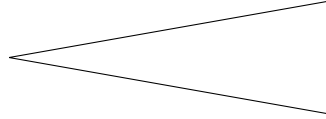
$$\frac{784}{25} = 31.36$$

regular nonagon with  
adjoined equilateral tri-  
angles



$$\approx 32.14624235$$

isosceles triangle,  $20^\circ$   
apex angle



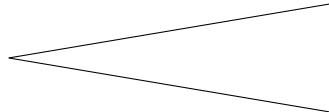
$$\approx 32.21915602$$

rectangle, 6:1



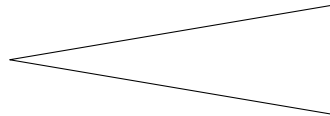
$$\frac{98}{3} = 32.\bar{6}$$

circular sector,  $1/3$  ra-  
dian



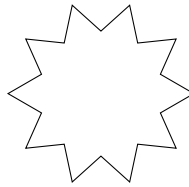
$$\frac{98}{3} = 32.\bar{6}$$

isosceles triangle, 3:1  
sides



$$\frac{196}{\sqrt{35}} \approx 33.13004679$$

regular decagon with  
adjoined equilateral tri-  
angles



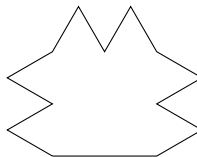
$$\approx 33.26587053$$

rectangle, 6:1, with two  
adjoined semicircles



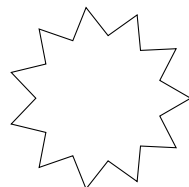
$$\frac{(24 + 2\pi)^2}{24 + \pi} \approx 33.78841190$$

square with two ad-  
joined equilateral trian-  
gles on three sides



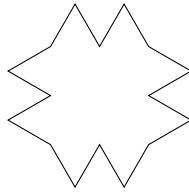
$$\frac{450}{8 + 3\sqrt{3}} \approx 34.10084891$$

regular 11-gon with ad-  
joined equilateral trian-  
gles



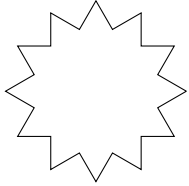
$$\approx 34.25632027$$

square with two ad-joined equilateral trian- gles on each side



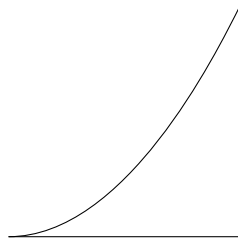
$$\frac{128}{2 + \sqrt{3}} \approx 34.29749663$$

regular 12-gon with ad-joined equilateral trian- gles



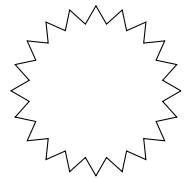
$$\approx 35.13843876$$

region bounded by  $y = x^2$ ,  $y = 0$ ,  $x = 1$



$$\approx 36.30913022$$

regular 20-gon with ad-joined equilateral trian- gles



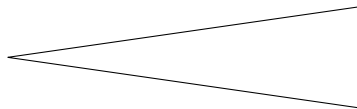
$$\approx 35.13843876$$

rectangle, 7:1



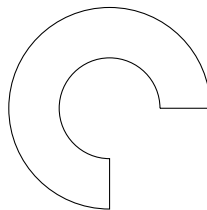
$$\frac{256}{7} \approx 36.57142857$$

circular sector,  $2/7$  ra- dian



$$\frac{256}{7} \approx 36.57142857$$

three-quarter circular sector with half-radius three-quarter circular sector removed



$$\frac{(9\pi + 4)^2}{9\pi} \approx 36.84021812$$

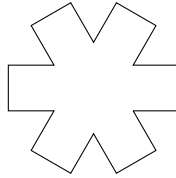


parabolic chunk: region bounded by  $y = x^2$  and  $y = 100$



$$\approx 36.99450714$$

regular hexagon with adjoined squares



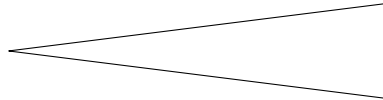
$$\approx 37.682848$$

rectangle, 8:1



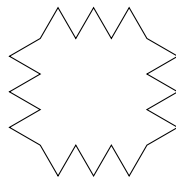
$$\frac{81}{2} = 40.5$$

circular sector, 1/4 radian



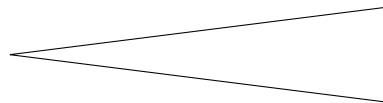
$$\frac{81}{2} = 40.5$$

square with three equilateral triangles adjoined to each side



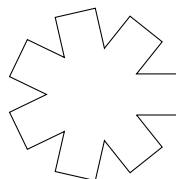
$$\frac{192}{3 + \sqrt{3}} = 40.57437416$$

isosceles triangle, 4:1 sides



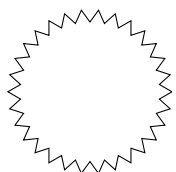
$$\frac{324}{\sqrt{63}} \approx 40.82016308$$

regular septagon with adjoined squares



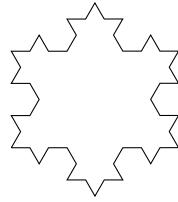
$$\approx 41.471095$$

regular 30-gon with adjoined equilateral triangles



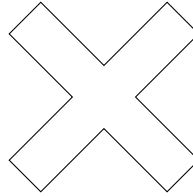
$$\approx 42.68026645$$

Koch curve, stage 3



$$\frac{384}{5\sqrt{3}} \approx 44.34050067$$

9-square cross



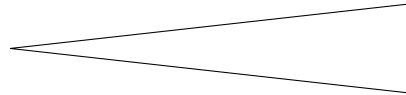
$$\frac{400}{9} = 44.\bar{4}$$

rectangle, 9:1



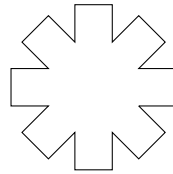
$$\frac{400}{9} = 44.\bar{4}$$

circular sector,  $2/9$  radian



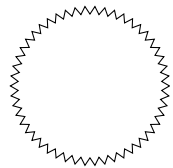
$$\frac{400}{9} = 44.\bar{4}$$

regular octagon with  
adjoined squares



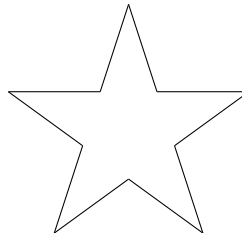
$$\approx 44.900282$$

regular 50-gon with ad-  
joined equilateral trian-  
gles



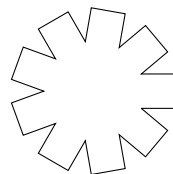
$$\approx 45.38596197$$

5 pointed star



$$20\sqrt{10 - 2\sqrt{5}} \approx 47.02282018$$

regular nonagon with  
adjoined squares



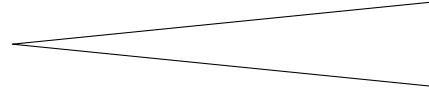
$$\approx 48.017945$$

rectangle, 10:1



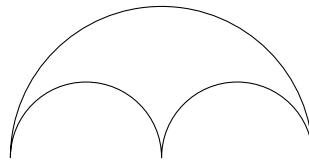
$$\frac{242}{5} = 48.4$$

circular sector,  $1/5$  radian



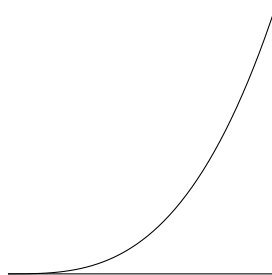
$$\frac{242}{5} = 48.4$$

semicircle augmented by two semicircles, III (arbelos)



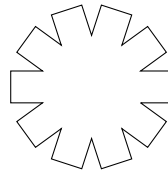
$$16\pi \approx 50.26548245$$

region bounded by  $y = x^3$ ,  $y = 0$  and  $x = 1$



$$\approx 50.34940281$$

regular decagon with adjoined squares



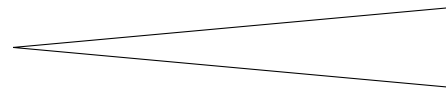
$$\approx 50.864099$$

rectangle, 11:1



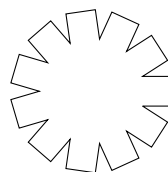
$$\frac{576}{11} = 52.\overline{36}$$

circular sector,  $2/11$  radian



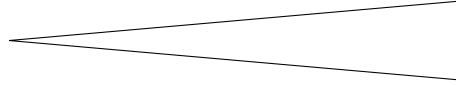
$$\frac{576}{11} = 52.\overline{36}$$

regular 11-gon with adjoined squares



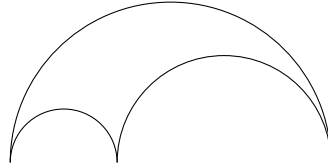
$$\approx 53.472417$$

isosceles triangle,  $10^\circ$   
apex angle



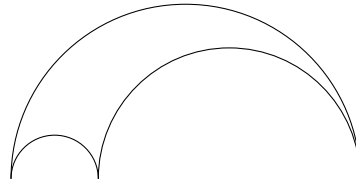
$$\approx 54.45067722$$

arbelos, one-third and  
two-thirds arcs



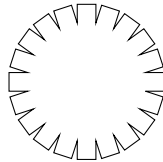
$$18\pi \approx 56.54866776$$

arbelos, one-quarter  
and three-quarter arcs



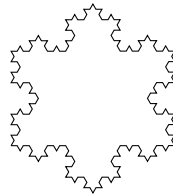
$$\frac{64\pi}{3} \approx 67.02064328$$

regular 20-gon with ad-  
joined squares



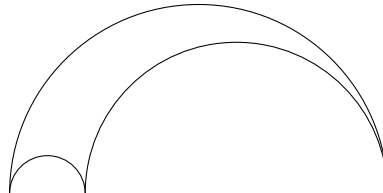
$$\approx 69.80970978$$

Koch curve, stage 4



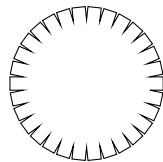
$$\frac{6144}{47\sqrt{3}} \approx 75.47319263$$

arbelos, one-fifth and  
four-fifths arcs



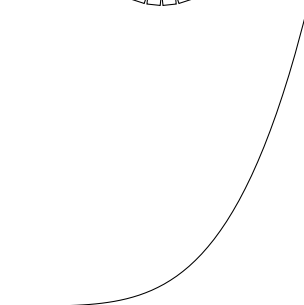
$$25\pi \approx 78.53981634$$

regular 30-gon with ad-  
joined squares



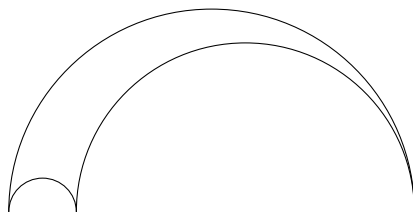
$$\approx 79.91496779$$

region bounded by  $y =$   
 $x^4$ ,  $y = 0$  and  $x = 1$



$$\approx 80.73353354$$

arbelos, one-sixth and five-sixths arcs



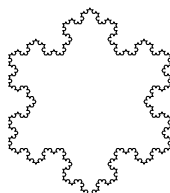
$$\frac{144\pi}{5} \approx 90.47786842$$

isosceles triangle,  $5^\circ$   
apex angle



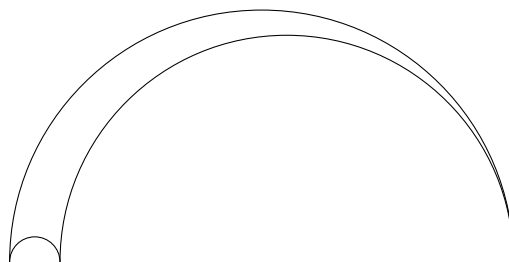
$$\approx 99.97197122$$

Koch curve, stage 5



$$\frac{98304}{431\sqrt{3}} \approx 131.68408553$$

arbelos, one-tenth and nine-tenths arcs



$$\frac{400\pi}{9} \approx 139.62634016$$

## General formulas

### Rectangles

For a rectangle with aspect ratio  $m$  (i.e., the ratio of non-equal length sides is  $m : 1$ ), the ratio is

$$\frac{(2m + 2)^2}{m} = 4m + 8 + \frac{4}{m}.$$

### Right triangles

For a right triangle with non-right angle  $\theta$ , the ratio is

$$\frac{2(1 + \sin \theta + \cos \theta)^2}{\sin \theta \cos \theta}.$$

### Isosceles triangles

For an isosceles triangle in which the sides with the same length are  $l$  times the length of the other side, the ratio is

$$\frac{4(2l + 1)^2}{\sqrt{4l^2 - 1}}.$$

For an isosceles triangle with an "apex" angle of  $\theta$ , the ratio is  $4 \tan \frac{\theta}{2} \left(1 + \csc \frac{\theta}{2}\right)^2$ .

### Rectangle with adjoined semicircles

For a rectangle with aspect ratio  $m$  with semicircles adjoined to the "1" sides, the ratio is

$$\frac{(4m + 2\pi)^2}{4m + \pi}.$$

For a value of  $m \approx 1.637129085772\dots$ , this is equal to the ratio for the  $m : 1$  rectangle. If  $m > 1.637129085772\dots$ , adjoining the semicircles decreases the ratio, while for smaller  $m$ , adjoining semicircles increases the ratio.

### Regular polygons

For a regular polygon with  $n$  sides, the ratio is  $4n \tan \frac{\pi}{n}$ .

### Circular sectors

For a circular sector with angle  $\theta$ , the ratio is

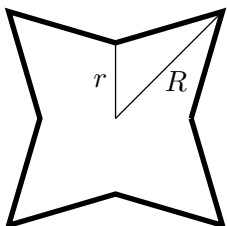
$$\frac{2(2 + \theta)^2}{\theta}$$

This has a minimum of 16 at  $\theta = 2$ .

### Polygonal Stars

For an  $n$ -sided polygonal star with equal side lengths, like the one shown, with inner radius  $r$  and outer radius  $R$ , the area is  $\frac{1}{2}nrR \sin \frac{2\pi}{n}$  and the perimeter is  $n\sqrt{r^2 + R^2 - 2rR \cos \frac{2\pi}{n}}$  so the isoperimetric ratio is

$$\frac{2n(r^2 + R^2 - 2rR \cos \frac{2\pi}{n})}{rR \sin \frac{2\pi}{n}}.$$



A special case of this are stars based on a regular polygon with  $m \geq 5$  sides by extending the sides of the polygon until they intersect. In this case, we have  $n = 2m$  and  $R = r(\cos \frac{\pi}{m} + \sin \frac{\pi}{m} \tan \frac{2\pi}{m})$ . In the table, such stars are called  $n$ -pointed stars.

### Regular polygons with adjoined equilateral triangles

If we adjoin equilateral triangles to the sides of a regular  $n$ -gon, the resulting figure has an isoperimetric ratio of

$$\frac{16n}{\sqrt{3} + \cot \frac{\pi}{n}}.$$

As  $n$  tends to infinity, this approaches  $16\pi = 50.265482\dots$  from below.

### Regular polygons with adjoined squares

If we adjoin squares to the sides of a regular  $n$ -gon, the resulting figure has an isoperimetric ratio of

$$\frac{36n}{4 + \cot \frac{\pi}{n}}$$

As  $n$  tends to infinity, this approaches  $36\pi = 113.097\dots$  from below.

### Regular polygons with adjoined semicircles

If we adjoin semicircles to the sides of a regular  $n$ -gon, the resulting figure has an isoperimetric ratio of

$$\frac{\pi^2 n}{\frac{\pi}{2} + \cot \frac{\pi}{n}}$$

As  $n$  tends to infinity, this approaches  $\pi^3 \approx 31.0063\dots$  from below.

### $n$ -square diamond/cross

The ratio is

$$\frac{(8m + 12)^2}{2m^2 + 6m + 5}$$

where  $m$  is the order of the diamond ( $m = 0$  is the 5-square cross,  $m = 1$  is the 13-square diamond, etc.)

This approaches 32 from below as  $m$  tends to infinity.

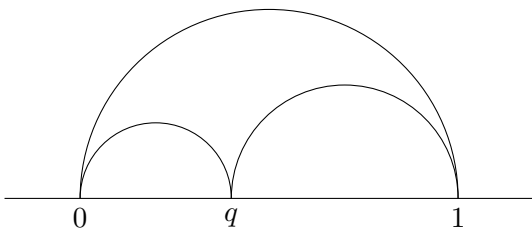
### Koch curve

The ratio is

$$\frac{180 \left(\frac{4}{3}\right)^{2n}}{\sqrt{3} \left(8 - 3 \left(\frac{4}{9}\right)^n\right)}$$

for the  $n$ -th iteration (i.e.,  $n = 0$  is an equilateral triangle,  $n = 1$  is a six-pointed star, etc.)

### Arbelos



The area is  $\frac{\pi}{4}q(1 - q)$  and the perimeter is  $\pi$ , so the ratio is

$$\frac{4\pi}{q(1 - q)}.$$

### Parabolic chunk

For regions bounded by a parabola and a line perpendicular to the parabola's axis, the ratio is

$$\frac{3 \left( 2a(2 + \sqrt{1 + 4a^2}) + \log(2a + \sqrt{1 + 4a^2}) \right)^2}{16a^3}$$

where the parabola is  $y = ax^2$  and the bounding line is  $y = a$  (we can express all parabolic chunks in this form; note, for instance, that the shape bounded by  $y = x^2$  and  $y = b$  is the same as the shape bounded by  $y = \sqrt{b}x^2$  and  $y = \sqrt{b}$ ).

### Aztec diamond

For an  $n$ -th order Aztec diamond, which looks like four copies of a set of  $n$  steps (of unit width and height) stuck together, the area is  $2n(n + 1)$  and the perimeter is  $8n$ , so the ratio is

$$\frac{P^2}{A} = \frac{64n^2}{2n(n + 1)} = \frac{32n}{n + 1}.$$

Incidentally, this is the same ratio as for a set of  $n$  unit steps themselves, as the perimeter is  $4n$  and the area is  $\frac{1}{2}n(n + 1)$  which yields

$$\frac{P^2}{A} = \frac{(4n)^2}{\frac{1}{2}n(n + 1)} = \frac{32n}{n + 1}.$$

This is because the "stair" part and the non-stair part have the same length, so putting four of them together exactly doubles the perimeter while quadrupling the area (unlike what happens when you put, say, four quarter-circles together to make a circle).

### Reuleaux polygons

For an  $n$ -sided Reuleaux polygon ( $n$  odd), the isoperimetric ratio is

$$\frac{4\pi^2 \sin^2 \left( \frac{n-1}{n} \pi \right)}{\frac{n}{2} \sin \frac{2\pi}{n} + 2\pi \sin^2 \frac{n-1}{2n} \pi - n \left( \sin \frac{\pi}{n} \right) \left( 1 + \cos \frac{\pi}{n} \right)}$$