

A Table of Isoperimetric Ratios

version August 16, 2025

Matthew M. Conroy

conroy "at" math dot washington dot edu

For a simple closed plane curve, the **isoperimetric ratio** is

$$\frac{P^2}{A}$$

where P is the length of the curve, and A is the area enclosed by the curve. This ratio is a dimensionless constant that is invariant under scaling: changing the *size* of the shape does not change this ratio.

The table below gives this ratio for a variety of shapes.

One can prove that the circle is the plane curve with the smallest (and hence, smallest possible) isoperimetric ratio, and so the table begins with the circle.

I started this table as a result of a homework problem in our precalculus class at the University of Washington. The problem asks the student to consider a situation in which we have a piece of wire of known length; we want to cut the wire and bend one resulting piece into a circle and the other piece into a square. The question is: where should we cut the wire so that the area of the circle plus the area of the square is minimized?

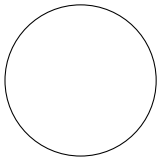
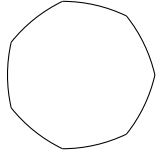
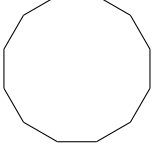
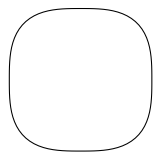
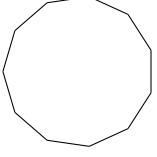
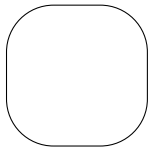
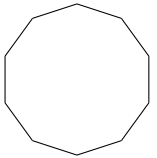
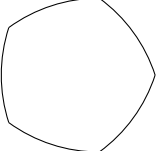
When writing exam questions of a similar flavor, the relative “efficiencies” of different planar shapes come under consideration.

For any wire length, if the two shapes have isoperimetric ratios r_1 and r_2 , then the wire should be cut into two pieces with lengths in the ratio $r_1 : r_2$ if we wish to minimize the total area.

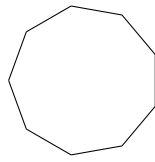
Cited by

This table is cited in the following paper:

Hirvonen, Petri & Boissonière, Gabriel & Fan, Zheyong & Achim, Cristian & Provas, Nikolas & Elder, Ken & Ala-Nissila, Tapio. (2018). Grain extraction and microstructural analysis method for two-dimensional poly and quasicrystalline solids. *Physical Review Materials*. 2. 10.1103/PhysRevMaterials.2.103603.

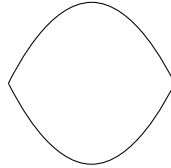
circle		$4\pi \approx 12.56637061$
Reuleaux heptagon		≈ 12.78538634
regular 12-gon		$96 - 48\sqrt{3} \approx 12.86156124$
super ellipse $ x ^3 + y ^3 = 1$		≈ 12.87612775
regular 11-gon		$44 \tan \frac{\pi}{11} \approx 12.91956568$
square with quarter circle rounded corners (radius of corners is one-third of overall width)		$\frac{(4 + 2\pi)^2}{5 + \pi} \approx 12.98810988$
regular 10-gon		$40 \tan \frac{\pi}{10} \approx 12.99678784$
Reuleaux pentagon		≈ 13.01205315

regular 9-gon



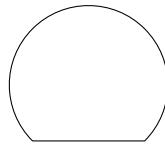
$$36 \tan \frac{\pi}{9} \approx 13.10292843$$

optimum region
bounded by symmetric
parabolic arcs (parabolas
are approximately
 $y = 0.97300151x^2$
on the interval
 $(-0.97300151, 0.97300151)$)



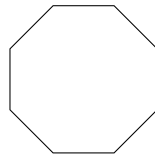
$$\approx 13.11934320$$

three-quarter truncated
circular arc



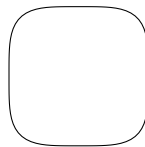
$$\approx 13.14170265$$

regular octagon (equiv-
alent to square aug-
mented by four opti-
mal isosceles triangles)



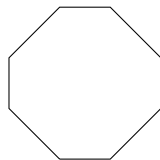
$$32(\sqrt{2} - 1) \approx 13.25483399$$

super ellipse
 $|x|^4 + |y|^4 = 1$



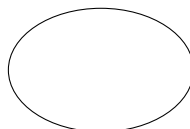
$$\approx 13.28104122$$

square with hacked off
corners (horizontal and
vertical edges are one-
third overall width)

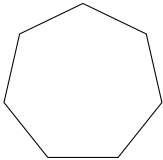
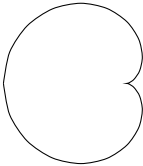
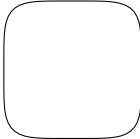
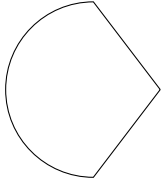
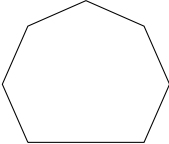
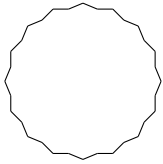
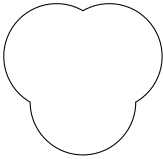
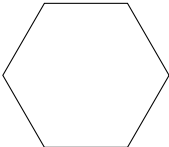


$$\frac{(4 + 4\sqrt{2})^2}{7} \approx 13.32211914$$

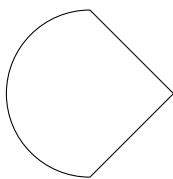
ellipse, 3:2



$$\approx 13.35374554$$

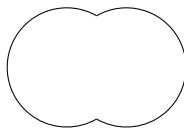
regular septagon		$28 \tan \frac{\pi}{7} \approx 13.48408932$
cardioid		$\frac{128}{3\pi} \approx 13.58122181$
super ellipse $ x ^5 + y ^5 = 1$		≈ 13.61890282
semicircle with optimal adjoined isosceles tri- angle		≈ 13.71361987
square with three opti- mal adjoined isosceles triangles		≈ 13.75752837
16 pointed star		≈ 13.77927647
equilateral triangle with three adjoined semicircles		≈ 13.78342300
regular hexagon		$\frac{24}{\sqrt{3}} \approx 13.85640646$

semicircle with adjoined right isosceles triangle



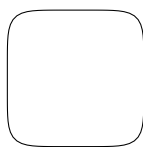
$$\frac{2(2\sqrt{2} + \pi)^2}{\pi + 2} \approx 13.86385058$$

two overlapping circles of same radius, centers on circles



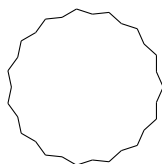
$$\frac{384\pi^2}{72\pi + 27\sqrt{3}} \approx 13.88455266$$

super ellipse
 $|x|^6 + |y|^6 = 1$



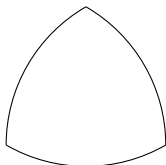
$$\approx 13.88994096$$

15 pointed star



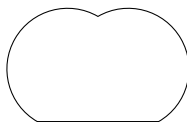
$$\approx 13.96032741$$

Reuleaux triangle



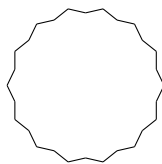
$$\frac{2\pi^2}{\pi - \sqrt{3}} \approx 14.00398921$$

equilateral triangle with two adjoined semicircles



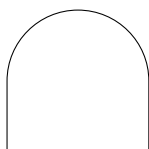
$$\frac{4(1 + \pi)^2}{\pi + \sqrt{3}} \approx 14.07800126$$

14 pointed star



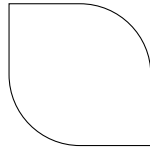
$$\approx 14.18654408$$

semicircle with optimal adjoined rectangle



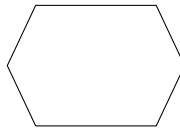
$$8 + 2\pi \approx 14.2831853071$$

square with two equally, optimally rounded corners (radii are half the side length)



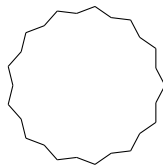
$$8 + 2\pi \approx 14.2831853071$$

square with two optimal adjoined isosceles triangles



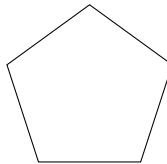
$$\approx 14.35072219$$

13 pointed star



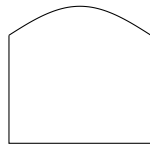
$$\approx 14.47485646$$

regular pentagon



$$\frac{100}{\sqrt{25 + 10\sqrt{5}}} \approx 14.53085056$$

optimal region bounded by $\frac{1}{x^2 + 1}$, the x -axis, and $x = \pm a$, where $a \approx 0.5181167$



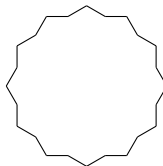
$$\approx 14.695335322$$

square with two adjoined semicircles



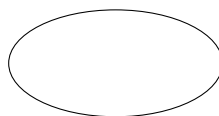
$$\frac{4(2 + \pi)^2}{4 + \pi} \approx 14.80676722$$

12 pointed star



$$\approx 14.85125168$$

ellipse, 2:1



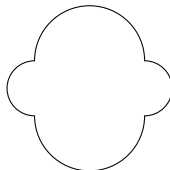
$$\approx 14.93924249$$

square with three ad-
joined semicircles



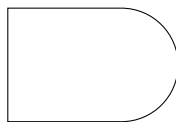
$$\frac{2(2 + 3\pi)^2}{8 + 3\pi} \approx 14.98160283$$

rectangle, 2:1, with four
adjoined semicircles



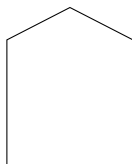
$$\frac{36\pi^2}{8 + 5\pi} \approx 14.98676855$$

square with adjoined
semicircle



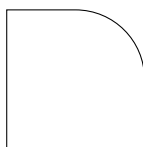
$$\frac{2(6 + \pi)^2}{8 + \pi} \approx 15.00121550$$

square with optimal
adjoined isosceles
triangle



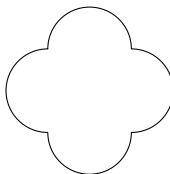
$$\approx 15.07344594$$

square with single op-
timally rounded corner
(radius is one half the
side length)



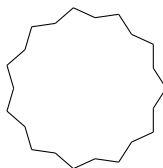
$$12 + \pi \approx 15.14159265$$

square with four ad-
joined semicircles



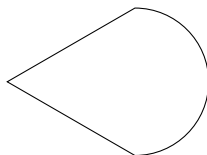
$$\frac{8\pi^2}{2 + \pi} \approx 15.35649370$$

11 pointed star



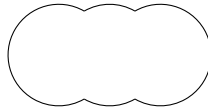
$$\approx 15.35751730$$

equilateral triangle
with adjoined semicir-
cle



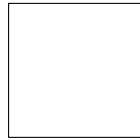
$$\frac{2(4 + \pi)^2}{\pi + 2\sqrt{3}} \approx 15.44193344$$

three overlapping circles with identical radii, centers on circles



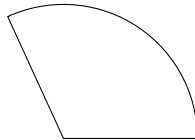
$$\approx 15.73789670$$

square



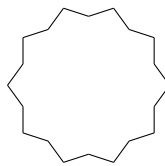
$$16$$

circular sector, two radians (optimal circular sector)



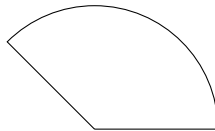
$$16$$

10 pointed star



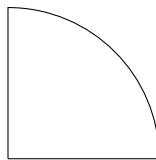
$$\approx 16.06491327$$

circular sector, $3\pi/4$ radians



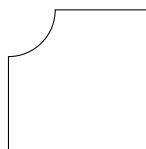
$$\frac{(8 + 3\pi)^2}{6\pi} \approx 16.10769443$$

quarter circular sector



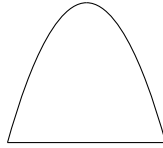
$$\frac{4(2 + \frac{\pi}{2})^2}{\pi} \approx 16.23455083$$

square with quarter-circle removed from one corner (radius one-third of square width)



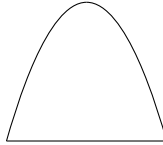
$$\frac{(20 + \pi)^2}{36 - \pi} \approx 16.29821265$$

optimal parabolic chunk (region bounded by $y = x^2$ and $y \approx 3.127903$)



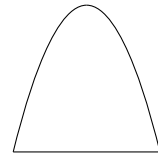
$$\approx 16.48263518$$

parabolic chunk: region bounded by $y = x^2$ and $y = 3$



$$\approx 16.48509413$$

parabolic chunk: region bounded by $y = x^2$ and $y = 4$



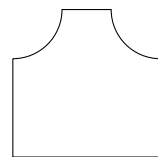
$$\approx 16.56740039$$

rectangle, 3:2 sides



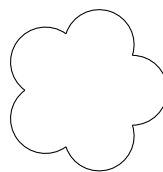
$$\frac{50}{3} = 16.\bar{6}$$

square with quarter-circle removed from two corners (radius one-third of square width)



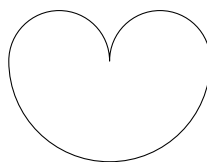
$$\frac{2(8 + \pi)^2}{18 - \pi} \approx 16.70907035$$

regular pentagon with adjoined semicircles



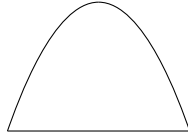
$$\frac{50\pi^2}{5\pi + 2\sqrt{25 + 10\sqrt{5}}} \approx 16.74415928$$

semicircle augmented by two semicircles, I



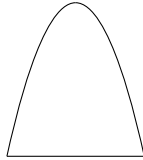
$$\frac{16}{3}\pi \approx 16.75516081$$

parabolic chunk: region bounded by $y = x^2$ and $y = 2$



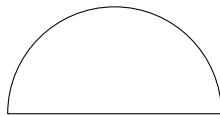
$$\approx 16.76938845$$

parabolic chunk: region bounded by $y = x^2$ and $y = 5$



$$\approx 16.79033103$$

semicircle



$$\frac{2(2 + \pi)^2}{\pi} \approx 16.82966439$$

golden rectangle (rectangle with side ratio $1:\phi$ where $\phi = \frac{1}{2}(1 + \sqrt{5})$)



$$8\phi + 4 \approx 16.944271909$$

rectangle with side ratio $1:1.63712909\dots$



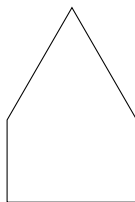
$$\approx 16.99181787$$

rectangle with side ratio $1:1.63712909\dots$ with adjoined semicircles



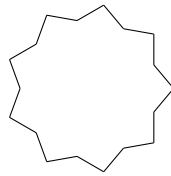
$$\approx 16.99181787$$

equilateral triangle with adjoined optimal rectangle



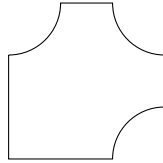
$$4(6 - \sqrt{3}) \approx 17.07179676$$

9 pointed star



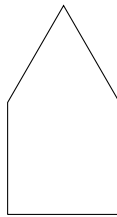
$$\approx 17.10465828$$

square with quarter-circle removed from three corners (radius one-third of square width)



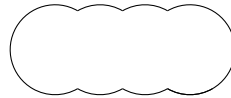
$$\frac{(12 + 3\pi)^2}{36 - 3\pi} \approx 17.27252212$$

square with adjoined equilateral triangle



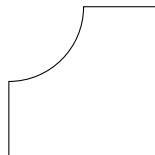
$$\frac{100}{4 + \sqrt{3}} \approx 17.44576302$$

four overlapping circles with identical radii, centers on circles



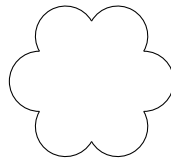
$$\approx 17.78054503$$

square with half-width radius quarter-circle removed from one corner



$$\frac{(12 + \pi)^2}{16 - \pi} \approx 17.83018860$$

regular hexagon with adjoined semicircles



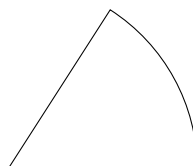
$$\frac{12\pi^2}{\pi + 2\sqrt{3}} \approx 17.92926648$$

rectangle, 2:1



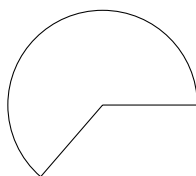
$$18$$

circular sector, one radian



$$18$$

circular sector, four radians



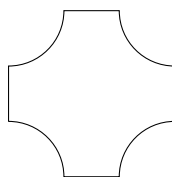
18

rectangle, 2:1, with one adjoined semicircle



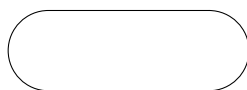
$$\frac{2(10 + \pi)^2}{16 + \pi} \approx 18.04462780$$

square with quarter-circles removed from each corner (radius 1/3 of square side)



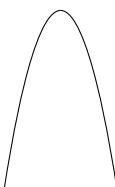
$$\frac{(4 + 2\pi)^2}{9 - \pi} \approx 18.04993982$$

rectangle, 2:1, with two adjoined semicircles



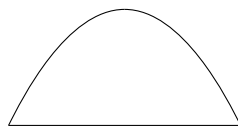
$$\frac{4(4 + \pi)^2}{8 + \pi} \approx 18.31061221$$

parabolic chunk: region bounded by $y = x^2$ and $y = 10$



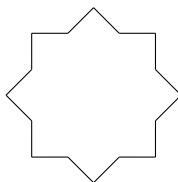
≈ 18.38876821

parabolic chunk: region bounded by $y = x^2$ and $y = 1$



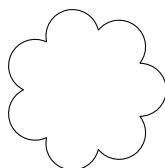
≈ 18.43547307

8 pointed star



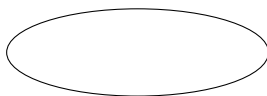
$$64 - 4^{11/4} \approx 18.74516600$$

regular septagon with adjoined semicircles



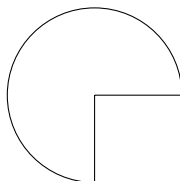
≈ 18.94192830

ellipse, 3:1



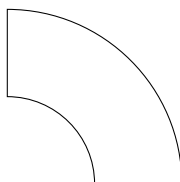
$$\approx 18.95220996$$

three-quarter circular sector



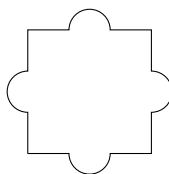
$$\frac{(4 + 3\pi)^2}{3\pi} \approx 19.12243068$$

quarter circular sector with half-radius quarter circular sector removed



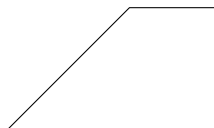
$$\frac{(4 + 3\pi)^2}{3\pi} \approx 19.12243068$$

square with an adjoined semicircle on each side's middle third



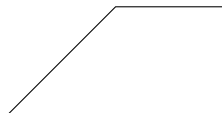
$$\frac{8(\pi + 4)^2}{18 + \pi} \approx 19.29933907$$

isosceles right triangle with optimal adjoined rectangle (1 : $\frac{\sqrt{2}}{2}$)



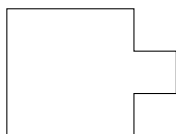
$$8 + 8\sqrt{2} \approx 19.31370849$$

isosceles right triangle with adjoined square



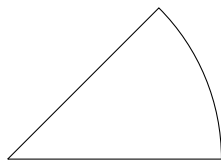
$$\frac{2(4 + \sqrt{2})^2}{3} \approx 19.54247233$$

square with square adjoined to middle-third of one side



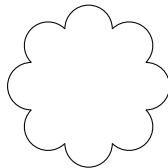
$$\frac{98}{5} = 19.6$$

circular sector, $\pi/4$ radians



$$\frac{(8 + \pi)^2}{2\pi} \approx 19.756712684$$

regular octagon with adjoined semicircles



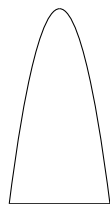
$$\approx 19.81346029$$

five overlapping circles with identical radii, centers on circles



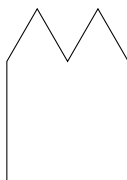
$$\approx 19.91184034$$

parabolic chunk: region bounded by $y = x^2$ and $y = 15$



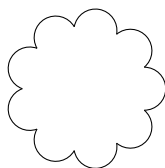
$$\approx 20.00749094$$

square with two equilateral triangles adjoined to one side



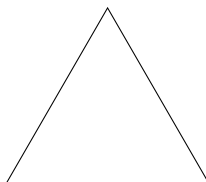
$$\frac{200}{8 + \sqrt{3}} \approx 20.55065309$$

regular nonagon with adjoined semicircles



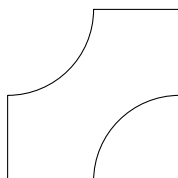
$$\approx 20.569895$$

equilateral triangle



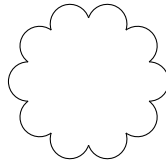
$$\frac{36}{\sqrt{3}} \approx 20.78460969$$

square with two half-width quarter-circles removed



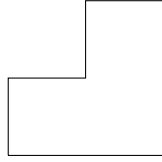
$$\frac{2(4 + \pi)^2}{8 - \pi} \approx 20.99549996$$

regular decagon with
adjoined semicircles



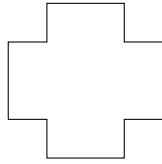
$$\approx 21.231897$$

two step unit staircase



$$\frac{64}{3} = 21.\bar{3}$$

square with quarter-
sized squares removed
from each corner
(aka order-2 Aztec
diamond)



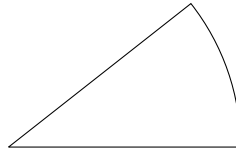
$$\frac{64}{3} = 21.\bar{3}$$

rectangle, 3:1



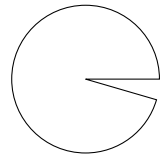
$$\frac{64}{3} = 21.\bar{3}$$

circular sector, $2/3$ radi-
ans



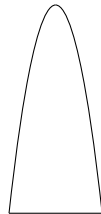
$$\frac{64}{3} = 21.\bar{3}$$

circular sector, 6 radi-
ans



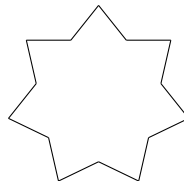
$$\frac{64}{3} = 21.\bar{3}$$

parabolic chunk: re-
gion bounded by $y =$
 x^2 and $y = 20$



$$\approx 21.50079173$$

7 pointed star



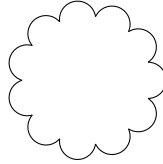
$$\approx 21.62680012$$

rectangle, 3:1, with one
adjoined semicircle



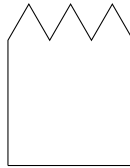
$$\frac{2(14 + \pi)^2}{24 + \pi} \approx 21.65194964$$

regular 11-gon with ad-
joined semicircles



$$\approx 21.81573534$$

square with three
equilateral triangles
adjoined to one side



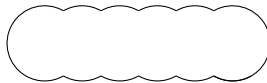
$$\frac{300}{12 + \sqrt{3}} \approx 21.84670041$$

rectangle, 3:1, with two
adjoined semicircles



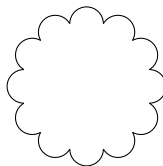
$$\frac{4(6 + \pi)^2}{12 + \pi} \approx 22.076598718$$

six overlapping circles
with identical radii,
centers on circles



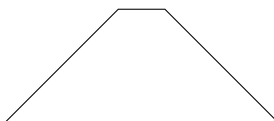
$$\approx 22.091743$$

regular 12-gon with ad-
joined semicircles



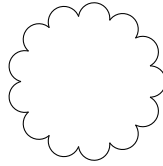
$$\approx 22.33427625$$

optimal "45 degree"
trapezoid (base= 1,
height= $\frac{1}{\sqrt{2+1}}$) (aka
optimal truncated
isosceles right tri-
angle aka optimal
rectangle with two
adjoined isosceles right
traingles)



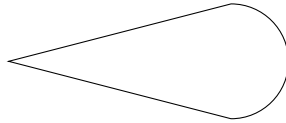
$$16\sqrt{2} \approx 22.62741699796$$

regular 13-gon with adjoined semicircles



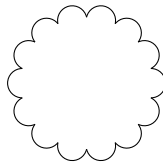
$$\approx 22.79777267$$

isosceles triangle, 2:1 sides, with adjoined semicircle



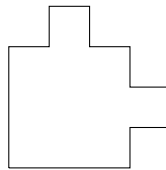
$$\frac{2(8 + \pi)^2}{\pi + 2\sqrt{15}} \approx 22.80310635$$

regular 14-gon with adjoined semicircles



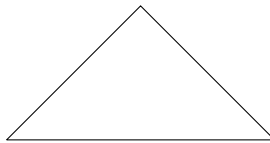
$$\approx 23.21447316$$

square with squares adjoined to middle third of two sides



$$\frac{256}{11} = 23.\overline{27}$$

isosceles right triangle



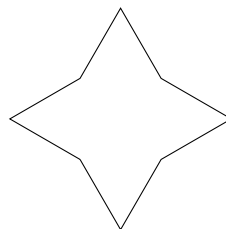
$$2(2 + \sqrt{2})^2 \approx 23.31370849$$

square with two adjoined isosceles right triangles



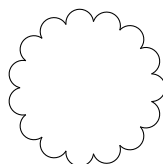
$$2(2 + \sqrt{2})^2 \approx 23.31370849$$

square with four adjoined equilateral triangles



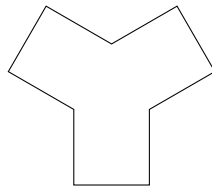
$$\frac{64}{1 + \sqrt{3}} \approx 23.42562584$$

regular 15-gon with adjoined semicircles



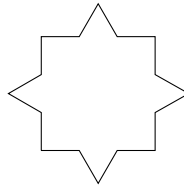
$$\approx 23.59107664$$

equilateral triangle with three adjoined squares



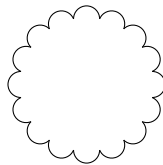
$$\frac{324}{12 + \sqrt{3}} \approx 23.59443644$$

square with an adjoined equilateral triangle on each side's middle third



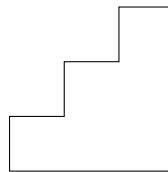
$$\frac{256}{9 + \sqrt{3}} \approx 23.85378196$$

regular 16-gon with adjoined semicircles



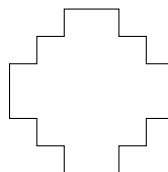
$$\approx 23.93307364$$

three-step unit staircase



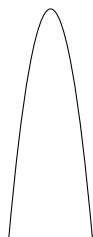
$$24$$

order-3 Aztec diamond



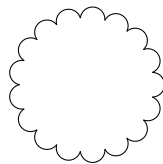
$$24$$

parabolic chunk: region bounded by $y = x^2$ and $y = 30$



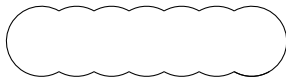
$$\approx 24.15608124$$

regular 17-gon with adjoined semicircles



$$\approx 24.2450034$$

seven overlapping circles with identical radii, centers on circles



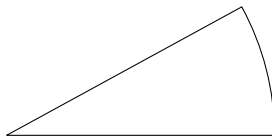
$$\approx 24.30117298$$

rectangle, 4:1



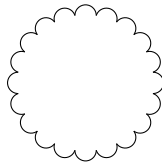
$$25$$

circular sector, 1/2 radian



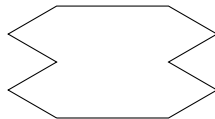
$$25$$

regular 20-gon with adjoined semicircles



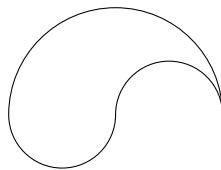
$$\approx 25.03530855$$

square with two equilateral triangles adjoined to two sides



$$\frac{144}{4 + \sqrt{3}} \approx 25.1218987469293$$

semicircle augmented with two semicircles, II



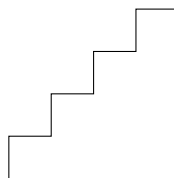
$$8\pi \approx 25.13274122$$

rectangle, 4:1, with one adjoined semicircle



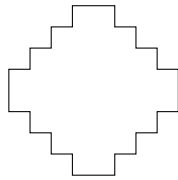
$$\frac{2(18 + \pi)^2}{32 + \pi} \approx 25.43805821$$

four-step unit staircase



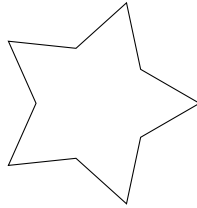
$$\frac{128}{5} = 25.6$$

order-4 Aztec diamond



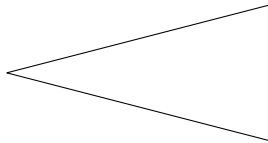
$$\frac{128}{5} = 25.6$$

regular pentagon with
adjoined equilateral tri-
angles



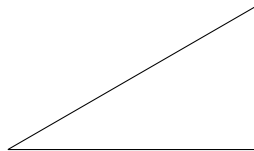
$$\approx 25.73644244$$

isosceles triangle, 2:1
sides



$$\frac{100}{\sqrt{15}} \approx 25.81988897$$

right triangle, 30-60-90



$$12 + 8\sqrt{3} \approx 25.85640646$$

rectangle, 4:1, with two
adjoined semicircles



$$\frac{4(8 + \pi)^2}{16 + \pi} \approx 25.94038836893$$

parabolic chunk: re-
gion bounded by $y =$
 x^2 and $y = 40$



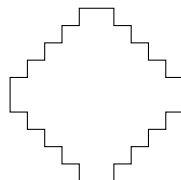
$$\approx 26.48326571$$

eight overlapping
circles with identical
radii, centers on circles



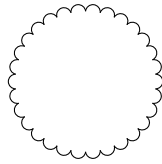
$$\approx 26.52987882$$

order-5 Aztec diamond



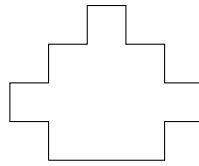
$$\frac{80}{3} = 26.\bar{6}$$

regular 30-gon with ad-joined semicircles



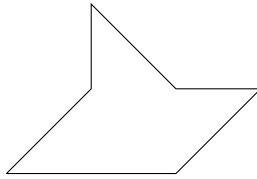
$$\approx 26.71031462$$

square with squares ad-joined to the middle thirds of three sides



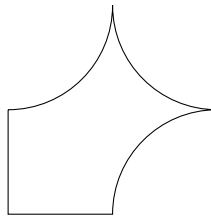
$$27$$

square with three ad-joined right isosceles triangles



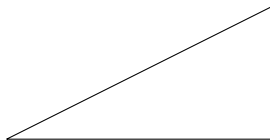
$$\frac{68 + 48\sqrt{2}}{5} \approx 27.17645019$$

square with quarter-circles removed from three corners



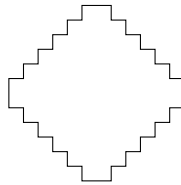
$$\frac{(4 + 3\pi)^2}{16 - 3\pi} \approx 27.40966954$$

right triangle, 2:1 legs



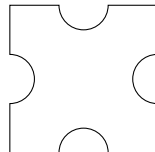
$$(3 + \sqrt{5})^2 = 14 + 6\sqrt{5} \approx 27.41640786$$

order-6 Aztec diamond



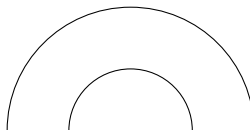
$$\frac{192}{7} = 27.\overline{428571}$$

square with one-third diameter semicircles removed from each side



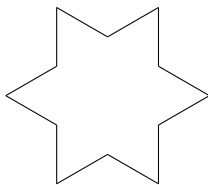
$$\frac{2(8 + 2\pi)^2}{18 - \pi} \approx 27.46046434$$

semicircle with half-diameter semicircle removed



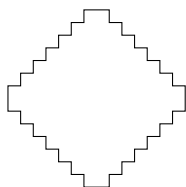
$$\frac{2(3\pi + 2)^2}{3\pi} \approx 27.69838228$$

6 pointed star (Koch curve, stage 2; regular hexagon with adjoined equilateral triangles)



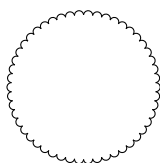
$$\sqrt{768} \approx 27.71281292$$

order-7 Aztec diamond



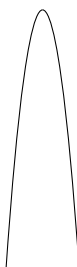
$$28$$

regular 50-gon with adjoined semicircles



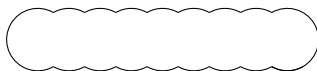
$$\approx 28.25482853$$

parabolic chunk: region bounded by $y = x^2$ and $y = 50$



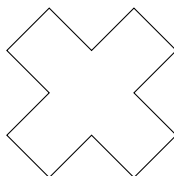
$$\approx 28.57579596$$

nine overlapping circles with identical radii, centers on circles



$$\approx 28.77186347$$

5-square cross



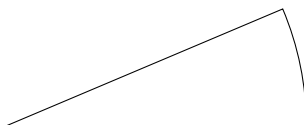
$$\frac{144}{5} = 28.8$$

rectangle, 5:1



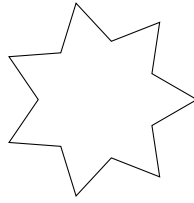
$$\frac{144}{5} = 28.8$$

circular sector, $2/5$ radian



$$\frac{144}{5} = 28.8$$

regular septagon with
adjoined equilateral tri-
angles



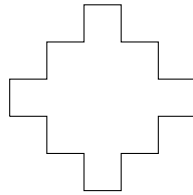
$$\approx 29.40734584$$

rectangle, 5:1, with two
adjoined semicircles



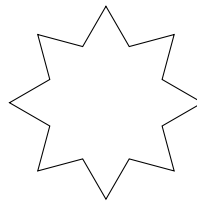
$$\frac{4(10 + \pi)^2}{20 + \pi} \approx 29.851265651086$$

13-square diamond



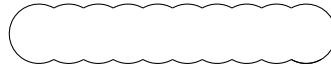
$$\frac{400}{13} \approx 30.76923076$$

regular octagon with
adjoined equilateral tri-
angles



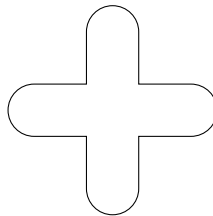
$$\approx 30.87116222$$

ten overlapping circles
with identical radii,
centers on circles



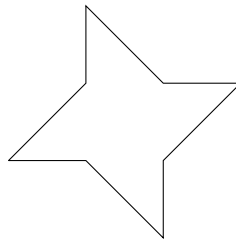
$$\approx 31.02338364$$

5-square cross with
four adjoined semicir-
cles



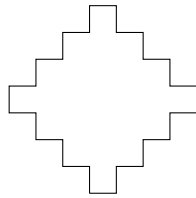
$$\frac{8(4 + \pi)^2}{\pi + 10} \approx 31.04789318$$

square with four ad-
joined isosceles right
triangles



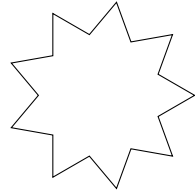
$$16 + \frac{32}{3}\sqrt{2} \approx 31.08494466$$

25-square diamond



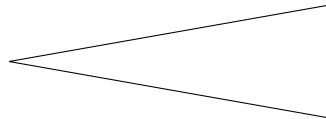
$$\frac{784}{25} = 31.36$$

regular nonagon with
adjoined equilateral tri-
angles



$$\approx 32.14624235$$

isosceles triangle, 20°
apex angle



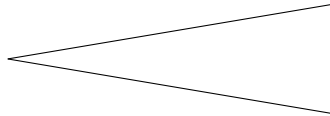
$$\approx 32.21915602$$

rectangle, 6:1



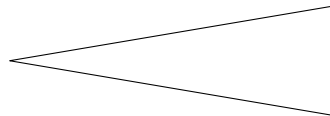
$$\frac{98}{3} = 32.\bar{6}$$

circular sector, $1/3$ ra-
dian



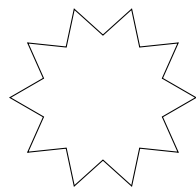
$$\frac{98}{3} = 32.\bar{6}$$

isosceles triangle, 3:1
sides



$$\frac{196}{\sqrt{35}} \approx 33.13004679$$

regular decagon with
adjoined equilateral tri-
angles



$$\approx 33.26587053$$

eleven overlapping
circles with identical
radii, centers on circles



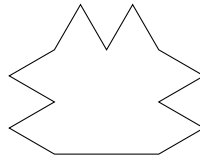
$$\approx 33.28198214$$

rectangle, 6:1, with two
adjoined semicircles



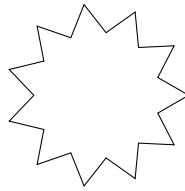
$$\frac{(24 + 2\pi)^2}{24 + \pi} \approx 33.78841190$$

square with two ad-
joined equilateral trian-
gles on three sides



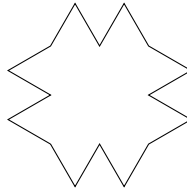
$$\frac{450}{8 + 3\sqrt{3}} \approx 34.10084891$$

regular 11-gon with ad-
joined equilateral trian-
gles



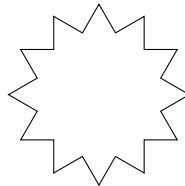
$$\approx 34.25632027$$

square with two ad-
joined equilateral trian-
gles on each side



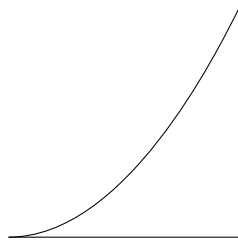
$$\frac{128}{2 + \sqrt{3}} \approx 34.29749663$$

regular 12-gon with ad-
joined equilateral trian-
gles



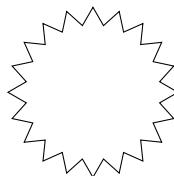
$$\approx 35.13843876$$

region bounded by $y =$
 x^2 , $y = 0$, $x = 1$



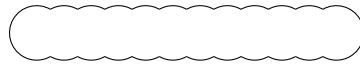
$$\approx 36.30913022$$

regular 20-gon with ad-
joined equilateral trian-
gles



$$\approx 35.13843876$$

twelve overlapping circles with identical radii, centers on circles



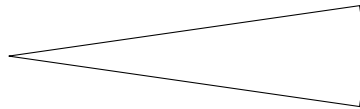
$$\approx 35.54597926$$

rectangle, 7:1



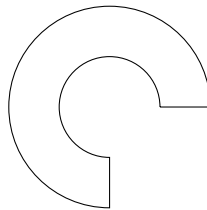
$$\frac{256}{7} \approx 36.57142857$$

circular sector, $2/7$ radian



$$\frac{256}{7} \approx 36.57142857$$

three-quarter circular sector with half-radius three-quarter circular sector removed



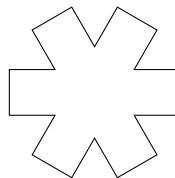
$$\frac{(9\pi + 4)^2}{9\pi} \approx 36.84021812$$

parabolic chunk: region bounded by $y = x^2$ and $y = 100$



$$\approx 36.99450714$$

regular hexagon with adjoining squares



$$\approx 37.682848$$

rectangle, 7:1, with two adjoining semicircles



$$\frac{(28 + 2\pi)^2}{28 + \pi} \approx 37.74170473$$

thirteen overlapping circles with identical radii, centers on circles



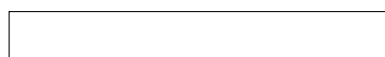
$$\approx 37.81418780$$

fourteen overlapping circles with identical radii, centers on circles



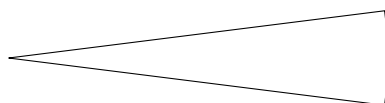
$$\approx 40.08574488$$

rectangle, 8:1



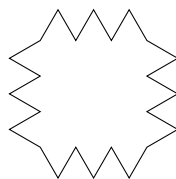
$$\frac{81}{2} = 40.5$$

circular sector, 1/4 radian



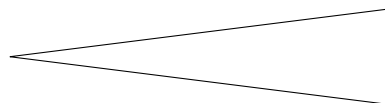
$$\frac{81}{2} = 40.5$$

square with three equilateral triangles adjoined to each side



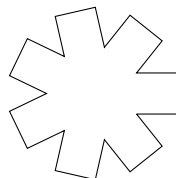
$$\frac{192}{3 + \sqrt{3}} = 40.57437416$$

isosceles triangle, 4:1 sides



$$\frac{324}{\sqrt{63}} \approx 40.82016308$$

regular septagon with adjoined squares



$$\approx 41.471095$$

rectangle, 8:1, with two adjoined semicircles



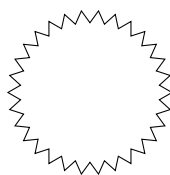
$$\frac{(32 + 2\pi)^2}{32 + \pi} \approx 41.70563046$$

fifteen overlapping circles with identical radii, centers on circles



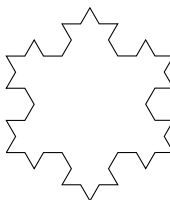
$$\approx 42.36000829$$

regular 30-gon with adjoined equilateral triangles



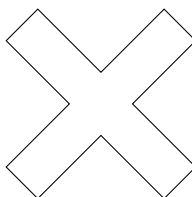
$$\approx 42.68026645$$

Koch curve, stage 3



$$\frac{384}{5\sqrt{3}} \approx 44.34050067$$

9-square cross



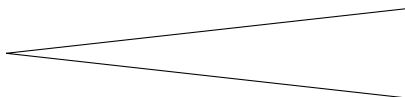
$$\frac{400}{9} = 44.\bar{4}$$

rectangle, 9:1



$$\frac{400}{9} = 44.\bar{4}$$

circular sector, $2/9$ radian



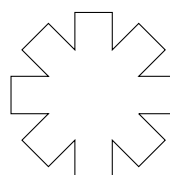
$$\frac{400}{9} = 44.\bar{4}$$

sixteen overlapping circles with identical radii, centers on circles



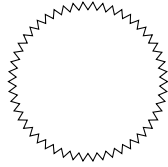
$$\approx 44.63649016$$

regular octagon with adjoined squares



$$\approx 44.900282$$

regular 50-gon with adjoined equilateral triangles



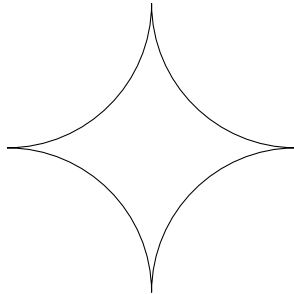
$$\approx 45.38596197$$

rectangle, 9:1, with two adjoined semicircles



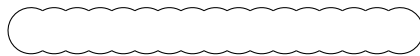
$$\frac{(36 + 2\pi)^2}{36 + \pi} \approx 45.67692928$$

four quarter-circle arcs



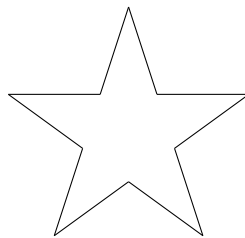
$$\frac{4\pi^2}{4 - \pi} \approx 45.99030724$$

seventeen overlapping circles with identical radii, centers on circles



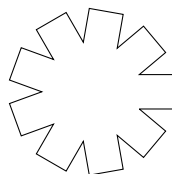
$$\approx 46.91481325$$

5 pointed star



$$20\sqrt{10 - 2\sqrt{5}} \approx 47.02282018$$

regular nonagon with adjoined squares



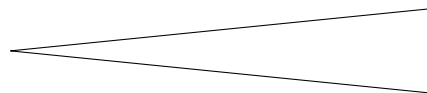
$$\approx 48.017945$$

rectangle, 10:1



$$\frac{242}{5} = 48.4$$

circular sector, 1/5 radian



$$\frac{242}{5} = 48.4$$

eighteen overlapping circles with identical radii, centers on circles



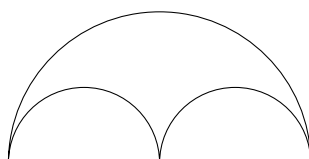
$$\approx 49.19468126$$

rectangle, 10:1, with two adjoined semicircles



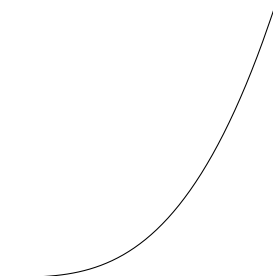
$$\frac{(40 + 2\pi)^2}{40 + \pi} \approx 49.65355033$$

semicircle augmented by two semicircles, III (arbelos)



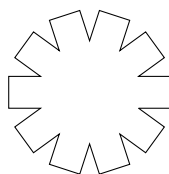
$$16\pi \approx 50.26548245$$

region bounded by $y = x^3$, $y = 0$ and $x = 1$



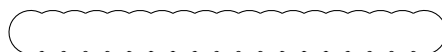
$$\approx 50.34940281$$

regular decagon with adjoined squares



$$\approx 50.864099$$

nineteen overlapping circles with identical radii, centers on circles



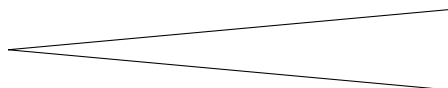
$$\approx 51.47585822$$

rectangle, 11:1



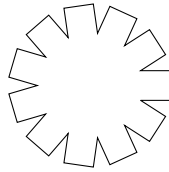
$$\frac{576}{11} = 52.\overline{36}$$

circular sector, $2/11$ radian



$$\frac{576}{11} = 52.\overline{36}$$

regular 11-gon with adjoined squares



$$\approx 53.472417$$

rectangle, 11:1, with two adjoined semicircles



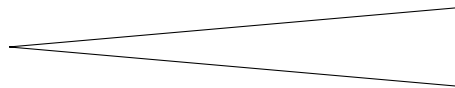
$$\frac{(44 + 2\pi)^2}{44 + \pi} \approx 53.63413882$$

twenty overlapping circles with identical radii, centers on circles



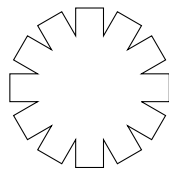
$$\approx 53.75815391$$

isosceles triangle, 10° apex angle



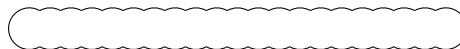
$$\approx 54.45067722$$

regular 12-gon with adjoined squares



$$\approx 55.87133488$$

twenty-one overlapping circles with identical radii, centers on circles



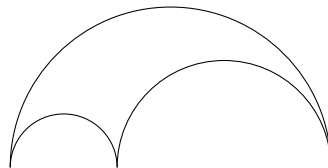
$$\approx 56.04141324$$

rectangle, 12:1



$$\frac{169}{3} = 56.\bar{3}$$

arbelos, one-third and two-thirds arcs



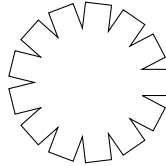
$$18\pi \approx 56.54866776$$

rectangle, 12:1, with two adjoined semicircles



$$\frac{(48 + 2\pi)^2}{48 + \pi} \approx 57.61776382$$

regular 13-gon with adjoined squares



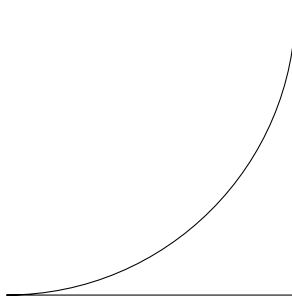
$$\approx 58.08498650$$

twenty-two overlapping circles with identical radii, centers on circles



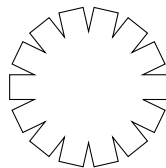
$$\approx 58.32550853$$

square with quarter circle removed



$$\frac{(4 + \pi)^2}{4 - \pi} \approx 59.41508520$$

regular 14-gon with adjoined squares



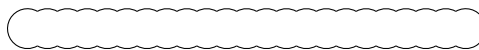
$$\approx 60.13396797$$

rectangle, 13:1



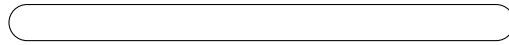
$$\frac{784}{13} = 60.\overline{307692}$$

twenty-three overlapping circles with identical radii, centers on circles



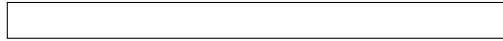
$$\approx 60.61033371$$

rectangle, 13:1, with two adjoined semicircles



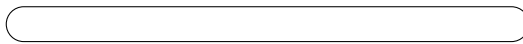
$$\frac{(52 + 2\pi)^2}{52 + \pi} \approx 61.60376452$$

rectangle, 14:1



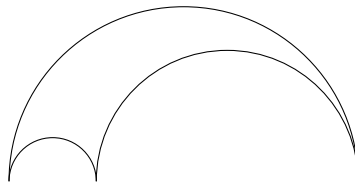
$$\frac{450}{7} = 64.\overline{285714}$$

rectangle, 14:1, with two adjoined semicircles



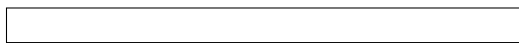
$$\frac{(56 + 2\pi)^2}{56 + \pi} \approx 65.59165889$$

arbelos, one-quarter and three-quarter arcs



$$\frac{64\pi}{3} \approx 67.02064328$$

rectangle, 15:1



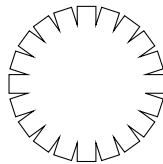
$$\frac{1024}{15} = 68.\overline{26}$$

rectangle, 15:1, with two adjoined semicircles



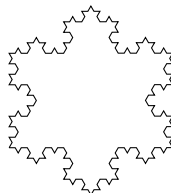
$$\frac{(60 + 2\pi)^2}{60 + \pi} \approx 69.58108704$$

regular 20-gon with adjoined squares



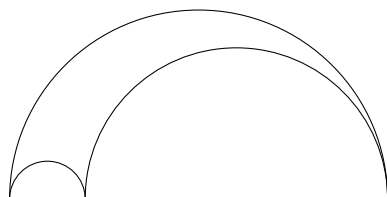
$$\approx 69.80970978$$

Koch curve, stage 4



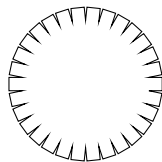
$$\frac{6144}{47\sqrt{3}} \approx 75.47319263$$

arbelos, one-fifth and four-fifths arcs



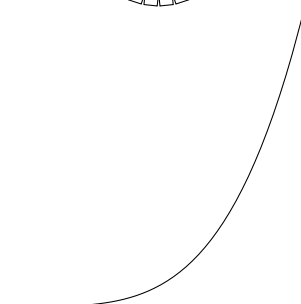
$$25\pi \approx 78.53981634$$

regular 30-gon with adjoined squares



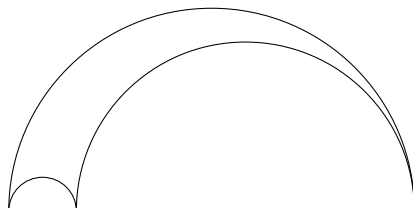
$$\approx 79.91496779$$

region bounded by $y = x^4$, $y = 0$ and $x = 1$



$$\approx 80.73353354$$

arbelos, one-sixth and five-sixths arcs



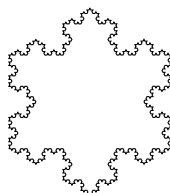
$$\frac{144\pi}{5} \approx 90.47786842$$

isosceles triangle, 5° apex angle



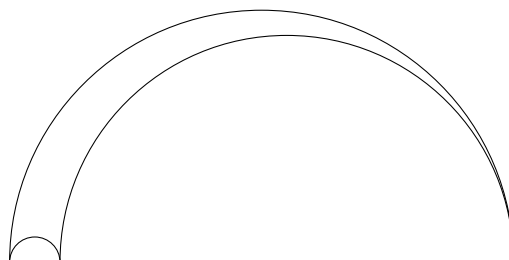
$$\approx 99.97197122$$

Koch curve, stage 5



$$\frac{98304}{431\sqrt{3}} \approx 131.68408553$$

arbelos, one-tenth and nine-tenths arcs



$$\frac{400\pi}{9} \approx 139.62634016$$

General formulas

Rectangles

For a rectangle with aspect ratio m (i.e., the ratio of non-equal length sides is $m : 1$), the ratio is

$$\frac{(2m + 2)^2}{m} = 4m + 8 + \frac{4}{m}.$$

Right triangles

For a right triangle with non-right angle θ , the ratio is

$$\frac{2(1 + \sin \theta + \cos \theta)^2}{\sin \theta \cos \theta}.$$

Isosceles triangles

For an isosceles triangle in which the sides with the same length are l times the length of the other side, the ratio is

$$\frac{4(2l + 1)^2}{\sqrt{4l^2 - 1}}.$$

For an isosceles triangle with an "apex" angle of θ , the ratio is $4 \tan \frac{\theta}{2} \left(1 + \csc \frac{\theta}{2}\right)^2$.

Rectangle with adjoined semicircles

For a rectangle with aspect ratio m with semicircles adjoined to the "1" sides, the ratio is

$$\frac{(4m + 2\pi)^2}{4m + \pi}.$$

For a value of $m \approx 1.637129085772\dots$, this is equal to the ratio for the $m : 1$ rectangle. If $m > 1.637129085772\dots$, adjoining the semicircles decreases the ratio, while for smaller m , adjoining semicircles increases the ratio.

Regular polygons

For a regular polygon with n sides, the ratio is $4n \tan \frac{\pi}{n}$.

Circular sectors

For a circular sector with angle θ , the ratio is

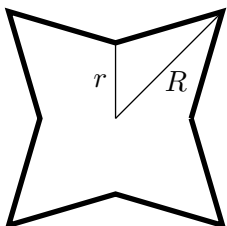
$$\frac{2(2 + \theta)^2}{\theta}$$

This has a minimum of 16 at $\theta = 2$.

Polygonal Stars

For an n -sided polygonal star with equal side lengths, like the one shown, with inner radius r and outer radius R , the area is $\frac{1}{2}nrR \sin \frac{2\pi}{n}$ and the perimeter is $n\sqrt{r^2 + R^2 - 2rR \cos \frac{2\pi}{n}}$ so the isoperimetric ratio is

$$\frac{2n(r^2 + R^2 - 2rR \cos \frac{2\pi}{n})}{rR \sin \frac{2\pi}{n}}.$$



A special case of this are stars based on a regular polygon with $m \geq 5$ sides by extending the sides of the polygon until they intersect. In this case, we have $n = 2m$ and $R = r(\cos \frac{\pi}{m} + \sin \frac{\pi}{m} \tan \frac{2\pi}{m})$. In the table, such stars are called n -pointed stars.

Regular polygons with adjoined equilateral triangles

If we adjoin equilateral triangles to the sides of a regular n -gon, the resulting figure has an isoperimetric ratio of

$$\frac{16n}{\sqrt{3} + \cot \frac{\pi}{n}}.$$

As n tends to infinity, this approaches $16\pi = 50.265482\dots$ from below.

Regular polygons with adjoined squares

If we adjoin squares to the sides of a regular n -gon, the resulting figure has an isoperimetric ratio of

$$\frac{36n}{4 + \cot \frac{\pi}{n}}$$

As n tends to infinity, this approaches $36\pi = 113.097\dots$ from below.

Regular polygons with adjoined semicircles

If we adjoin semicircles to the sides of a regular n -gon, the resulting figure has an isoperimetric ratio of

$$\frac{\pi^2 n}{\frac{\pi}{2} + \cot \frac{\pi}{n}}$$

As n tends to infinity, this approaches $\pi^3 \approx 31.0063\dots$ from below.

n -square diamond/cross

The ratio is

$$\frac{(8m + 12)^2}{2m^2 + 6m + 5}$$

where m is the order of the diamond ($m = 0$ is the 5-square cross, $m = 1$ is the 13-square diamond, etc.)

This approaches 32 from below as m tends to infinity.

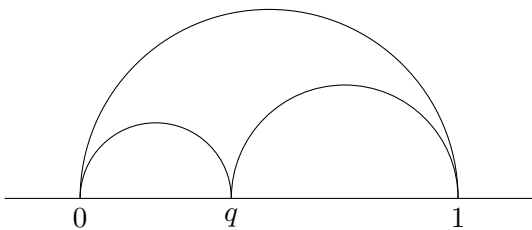
Koch curve

The ratio is

$$\frac{180 \left(\frac{4}{3}\right)^{2n}}{\sqrt{3} \left(8 - 3 \left(\frac{4}{9}\right)^n\right)}$$

for the n -th iteration (i.e., $n = 0$ is an equilateral triangle, $n = 1$ is a six-pointed star, etc.)

Arbelos



The area is $\frac{\pi}{4}q(1 - q)$ and the perimeter is π , so the ratio is

$$\frac{4\pi}{q(1 - q)}.$$

Parabolic chunk

For regions bounded by a parabola and a line perpendicular to the parabola's axis, the ratio is

$$\frac{3 \left(2a(2 + \sqrt{1 + 4a^2}) + \log(2a + \sqrt{1 + 4a^2}) \right)^2}{16a^3}$$

where the parabola is $y = ax^2$ and the bounding line is $y = a$ (we can express all parabolic chunks in this form; note, for instance, that the shape bounded by $y = x^2$ and $y = b$ is the same as the shape bounded by $y = \sqrt{b}x^2$ and $y = \sqrt{b}$).

Aztec diamond

For an n -th order Aztec diamond, which looks like four copies of a set of n steps (of unit width and height) stuck together, the area is $2n(n + 1)$ and the perimeter is $8n$, so the ratio is

$$\frac{P^2}{A} = \frac{64n^2}{2n(n + 1)} = \frac{32n}{n + 1}.$$

Incidentally, this is the same ratio as for a set of n unit steps themselves, as the perimeter is $4n$ and the area is $\frac{1}{2}n(n + 1)$ which yields

$$\frac{P^2}{A} = \frac{(4n)^2}{\frac{1}{2}n(n + 1)} = \frac{32n}{n + 1}.$$

This is because the "stair" part and the non-stair part have the same length, so putting four of them together exactly doubles the perimeter while quadrupling the area (unlike what happens when you put, say, four quarter-circles together to make a circle).

Reuleaux polygons

For an n -sided Reuleaux polygon (n odd), the isoperimetric ratio is

$$\frac{4\pi^2 \sin^2 \left(\frac{n-1}{n} \pi \right)}{\frac{n}{2} \sin \frac{2\pi}{n} + 2\pi \sin^2 \frac{n-1}{2n} \pi - n \left(\sin \frac{\pi}{n} \right) \left(1 + \cos \frac{\pi}{n} \right)}.$$

Overlapping circles

With two overlapping circles of radius 1, each center on the other circle, the overlapping area is

$$\alpha = \frac{2\pi}{\sqrt{3}} - \frac{\sqrt{3}}{2}$$

so the total area is

$$\frac{4\pi}{3} + \frac{\sqrt{3}}{2}$$

while the total perimeter is $\frac{8\pi}{3}$.

As we add circles in this way, the perimeter increases by $\frac{4\pi}{3}$ with each additional circle, while the area increases by

$$\pi - \alpha = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$

with each additional circle.

Thus, for n overlapping circles, the isoperimetric ratio is

$$\frac{\left(\frac{8\pi}{3} + (n-2)\left(\frac{2\pi}{3}\right) \right)^2}{\frac{4\pi}{3} + \frac{\sqrt{3}}{2} + (n-2)\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)} = \frac{8\pi^2(n+2)^2}{12\pi - 9\sqrt{3} + (6\pi + 9\sqrt{3})n}$$