Why A133899 is missing its 9th harmonic

The sequence A133899 is all positive integers congruent to 72,73,74,75,76,77,78,79 or 80 modulo 81.

As a result, the sound wave produced is periodic with period $81/s$ where s is the sampling rate.

Consider the waveform produced by the sequence of integers congruent to 72 mod 81. Call this $f(t)$, a function of time with period 81 time units (really, $81/s$, but leaving out the sampling rate simplifies the following nicely).

This function has a fourier expansion of the form

$$f(t) = \sum_{k=0}^{\infty} a_k \sin\left(\frac{2\pi k}{81} t + \phi\right)$$

That is, f is a sum of sinuosoids with frequencies that are multiples of $1/81$.

If we create the waveform f and look at its spectral lines, well see evenly spaced lines, one at every multiple of s/81 (about 544 hz with s=44100).

But, if we look at the waveform for A133899, we see an almost identical set of spectral lines, except that the 9th line is completely absent. Why is this?

The waveform of A133899 can be viewed as the sum of a bunch of shifted copies of f.

For instance, the waveform produced by the sequence of integers congruent to 73 mod 81 is

$$f(t-1)$$

and the waveform produced by the sequence of integers congruent to 73 mod 81 is

$$f(t-2)$$

etc. Hence, A133899's waveform can be expressed as the sum

$$A(t) = f(t) + f(t-1) + f(t-2) + \cdots + f(t-8)$$

Now, the 9th harmonic ($k = 9$) of f is

$$a_9 \sin\left(\frac{2\pi t}{9} + \phi\right)$$

so the 9th harmonic of A is

$$h_9 = a_9 \sum_{k=0}^{8} \sin\left(\frac{2\pi(t-k)}{9} + \phi\right)$$

We can show that this is identically zero by showing the summation is identically zero (i.e., it is zero for all t).

First, we use the fact, often call Euler's formula, that

$$e^{ix} = \cos x + i \sin x$$

so that

$$e^{i(2\pi(t-k)/9+\phi)} = \cos\left(\frac{2\pi}{9}(t-k) + \phi\right) + i \sin\left(\frac{2\pi}{9}(t-k) + \phi\right).$$

Hence, h_9 is the imaginary part of the sum of nine exponentials:

$$h_9 = \mathrm{Im} \sum_{k=0}^{8} e^{i(2\pi(t-k)/9+\phi)}$$

Simplifying this sum we have

$$\begin{aligned}
\sum_{k=0}^{8} e^{i(2\pi(t-k)/9+\phi)} &= \sum_{k=0}^{8} e^{\frac{2\pi i t}{9}} e^{-\frac{2\pi i k}{9}} e^{\phi} \\
&= e^{\frac{2\pi i t}{9}} e^{\phi} \sum_{k=0}^{8} e^{-\frac{2\pi i k}{9}} \\
&= e^{\frac{2\pi i t}{9}} e^{\phi} \sum_{k=0}^{8} \left(e^{-\frac{2\pi i}{9}}\right)^{k}
\end{aligned}$$

This last summation is a partial sum of a geometric series so we may us the general formula

$$\sum_{j=0}^{n} a^j = \frac{a^{n+1}-1}{a-1}$$

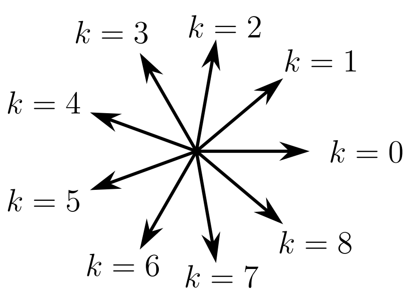
to find

$$\sum_{k=0}^{8} \left(e^{-\frac{2\pi i}{9}}\right)^{k} = \frac{(e^{-\frac{2\pi i}{9}})^9 - 1}{e^{-\frac{2\pi i}{9}} - 1} = \frac{e^{-2\pi i} - 1}{e^{-\frac{2\pi i}{9}-1}} = 0$$

since $e^{-2\pi i} = 1$.

Thus, all the shifted 9th harmonics perfectly cancel each other out.

One way to visualize what is happening is to think of the sines as the imaginary (or "y") components of vectors extending from the origin to the unit circle, like this:



Since the vectors are all of the same length, and are evenly spaced around the origin, we can see that the sum of all the vectors will be zero, as we showed.

In fact, if we ust consider the y-components, we can see that $k = 1$ cancels perfectly with $k = 8$, $k = 2$ cancels perfectly with $k = 7$, etc., leaving only $k = 0$ which has a zero y-component.