1. (a) There is an integer x such that, for every integer y, if $x + y \geq 4$, then $x^2y \geq 16$.
   negation: For all integers x, there exists an integer y such that $x + y \geq 4$ and $x^2y < 16$.

   (b) For every integer y, there exists an integer x such that, for every integer z, $xy = xz$.
   negation: There exists an integer y such that, for all integers x, there is an integer z such that $xy \neq yz$.

2. Let P be the (FALSE) statement: "Every positive integer is odd." Which of the following are logically equivalent to P? (Check all that apply.)

_______ If n is an odd integer, then n is positive.
___X___ If n is a positive integer, then n is odd.
_______ n is an odd integer only if n is a positive integer.
___X___ n is an odd integer if n is a positive integer.
_______ There exist integers that are both positive and odd.
___X___ No even integer is positive.
_______ n is an odd integer if and only if n is a positive integer.

3. Prove that, for all integers a, b, and c, with $c \neq 0$, $a \mid b$ if and only if $ca \mid cb$.

   Proof: Suppose a, b, and c are integers and $c \neq 0$.

   $\Rightarrow$ Suppose $a \mid b$.
   Then there is an integer k such that $b = ak$.
   Then $cb = c(ak) = (ca)k$ and thus $ca \mid cb$.

   $\Leftarrow$ Suppose $ca \mid cb$.
   Then there is an integer k such that $cb = (ca)k$.
   Then $0 = cb - c(ak) = c(b - ak)$, which implies that $c = 0$ or $b - ak = 0$.
   Since $c \neq 0$, it must be the case that $b - ak = 0$ or $b = ak$.
   Thus, $a \mid b$.

   Thus, $a \mid b$ if and only if $ca \mid cb$. □

4. (a) TRUE There are no integers m and n such that $14m + 21n = 5$.
   Proof: Suppose there exist integers m and n such that $14m + 21n = 5$.
   Then $5 = 7(2m + 3n)$, which implies that $7 \mid 5$.
   But this is a contradiction since the only divisors of 7 are 1 and 7.
   Thus, there do not exist integers m and n such that $14m + 21n = 5$. □

   (b) TRUE There exist integers a, b, and c such that $a|bc$ and $a \nmid b$ and $a \nmid c$.
   Proof: Let $a = 6$, $b = 3$ and $c = 4$.
   Then $bc = 12$.
   So, a, b, and c are integers and $a \mid bc$ but $a \nmid b$ and $a \nmid c$. □

(c) FALSE For all integers a and b, $|a+b| = |a| + |b|$.
Let $a = 5$ and $b = -2$.
Then $|a| = 5$, $|b| = 2$, and $a + b = 3$.
So $|a+b| = 3$ and $|a| + |b| = 5 + 2 = 7$ and thus $|a+b| \neq |a| + |b|$.

5. (a) Show that $28 \equiv 20$, $17 \equiv 1$, and $30 \equiv 2$.
$28 - 20 = 8 = 4(2)$. Since $2 \in \mathbb{Z}$, this implies that $28 \equiv 20$.
$17 - 1 = 16 = 4(4)$. Since $4 \in \mathbb{Z}$, this implies that $17 \equiv 1$.
$30 - 2 = 28 = 4(7)$. Since $7 \in \mathbb{Z}$, this implies that $30 \equiv 2$.

(b) Prove that, if $m \equiv 1$ and $n \equiv 1$, then $mn \equiv 1$.
Proof: Suppose that $m \equiv 1$ and $n \equiv 1$.
Then $m - 1 = 4k$ and $n - 1 = 4l$ for some $k, l \in \mathbb{Z}$.
This implies that $m = 4k + 1$ and $n = 4l + 1$.
We then have that $mn = (4k+1)(4l+1) = 16kl + 4k + 4l + 1 = 4(4kl + k + l) + 1$.
So, $mn - 1 = 4(4kl + k + l)$ and $4kl + k + l \in \mathbb{Z}$.
This means that $mn \equiv 1$. □

(c) Prove that, if $m \equiv 2$ and $n \equiv 2$, then $m + n \equiv 0$.
Proof: Suppose that $m \equiv 2$ and $n \equiv 2$.
Then $m - 2 = 4k$ and $n - 2 = 4l$ for some $k, l \in \mathbb{Z}$.
This implies that $m = 4k + 2$ and $n = 4l + 2$.
We then have that $(m+n) - 0 = m + n = (4k+2) + (4l+2) = 4k + 4l + 4 = 4(k+l+1)$.
Since $k + l + 1 \in \mathbb{Z}$, this implies that $m + n \equiv 0$. □