MATH 300

Sample Exam I

Name ______________________________

Student ID #______________

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam."

SIGNATURE:______________________________

1	8	
2	8	
3	10	
4	10	
5	14	
Total	50	

- Your exam should consist of this cover sheet, followed by 5 pages. Check that you have a complete exam.
- You are allowed to use the sheet of axioms and elementary properties of integers on the last page of this exam. You are not allowed to use any other sources.
- In your proofs, you may use any item on the sheet of axioms and elementary properties and any definition introduced in lecture. You may also use any facts you know about specific numbers (e.g., that $5 \cdot 7 = 35$ or that 3 divides 18 or that the only positive divisors of 10 are 1, 2, 5, and 10). All other claims should be justified.
- Turn your cell phone OFF and put it AWAY for the duration of the exam.

GOOD LUCK!

1. (8 points) Give a meaningful negation of each statement. You may use symbols like $\forall$ and $\exists$. Please put a box around your final answer.

   (a) There is an integer x such that, for every integer y, if $x + y \geq 4$, then $x^2 y \geq 16$.

   (b) For every integer y, there exists an integer x such that, for every integer z, $xy = xz$.

2. (8 points) Let P be the (FALSE) statement: "Every positive integer is odd." Which of the following are logically equivalent to P? (Check all that apply.)

_______ If n is an odd integer, then n is positive.
_______ If n is a positive integer, then n is odd.
_______ n is an odd integer only if n is a positive integer.
_______ n is an odd integer if n is a positive integer.
_______ There exist integers that are both positive and odd.
_______ No even integer is positive.
_______ n is an odd integer if and only if n is a positive integer.

3. (10 points) Prove that, for all integers a, b, and c, with $c \neq 0$, $a \mid b$ if and only if $ca \mid cb$.

4. (10 points) State whether each of the following is TRUE or FALSE. Justify your answer with a proof or a counterexample, whichever is appropriate.

   (a) ______ There are no integers m and n such that $14m + 21n = 5$.

   (b) ______ There exist integers a, b, and c such that $a|bc$ and $a \nmid b$ and $a \nmid c$.

   (c) ______ For all integers a and b, $|a + b| = |a| + |b|$.

5. (14 points) For this problem, we'll define the symbol $\equiv$ as follows.

   Definition: Let a and b be integers. We say that $a \equiv b$ if and only if there exists an integer k such that $a - b = 4k$.

   (a) Show that $28 \equiv 20$, $17 \equiv 1$, and $30 \equiv 2$.

   (b) Prove that, if $m \equiv 1$ and $n \equiv 1$, then $mn \equiv 1$.

   (c) Prove that, if $m \equiv 2$ and $n \equiv 2$, then $m + n \equiv 0$.

Axioms of the Integers (AIs)

Suppose a, b, and c are integers.

- **Closure:**

  $a + b$ and ab are integers.

- **Substitution of Equals:**

  If $a = b$, then $a+c = b+c$ and $ac = bc$.

- **Commutativity:**

  $a + b = b + a$ and $ab = ba$.

- **Associativity:**

  $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.

- **The Distributive Law:**

  $a(b + c) = ab + ac$

- **Identities:**

  $a + 0 = 0 + a = a$ and $a \cdot 1 = 1 \cdot a = a$

  0 is called the *additive identity*

  1 is called the *multiplicative identity.*

- **Additive Inverses:**

  There exists a real number $-a$ such that $a + (-a) = (-a) + a = 0$.

- **Trichotomy:**

  Exactly one of the following is true:
  $a > 0$, $-a > 0$, or $a = 0$.

Elementary Properties of the Integers (EPIs)

Suppose a, b, c, and d are integers.

(a) $a \cdot 0 = 0$

(b) If $a + c = b + c$, then $a = b$.

(c) $-a = (-1) \cdot a$

(d) $(-a) \cdot b = -(a \cdot b)$

(e) $(-a) \cdot (-b) = a \cdot b$

(f) If $a \cdot b = 0$, then $a = 0$ or $b = 0$.

(g) If $a \leq b$ and $b \leq a$, then $a = b$.

(h) If $a < b$ and $b < c$, then $a < c$.

(i) If $a < b$, then $a + c < b + c$.

(j) If $a < b$ and $0 < c$, then $ac < bc$.

(k) If $a < b$ and $c < 0$, then $bc < ac$.

(l) If $a < b$ and $c < d$, then $a + c < b + d$.

(m) If $0 \leq a < b$ and $0 \leq c < d$, then $ac < bd$.

(n) If $a < b$, then $-b < -a$.

(o) $0 \leq a^2$

(p) If $ab = 1$, then either $a = b = 1$ or $a = b = -1$.

NOTE: Properties (h)-(n) hold if each $<$ is replaced with $\leq$.