MATH 300 A Winter 2017
Exam I Answers

1. (6 points) Give a meaningful negation of each statement.

   (a) There exists an integer m such that, for every integer n, $m \mid n$ or $m > n$.
   NEGATION: For all integers m, there exists an integer n such that $m \nmid n$ and $m \leq n$.

   (b) For every $p > 0$, there exists $d > 0$ such that, for every real number x, if $|x - 3| < d$, then $|x^2 - 9| < p$.
   NEGATION: There exists a $p > 0$ such that, for all $d > 0$, there exists a real number x such that $|x - 3| < d$ and $|x^2 - 9| \geq p$.

2. (4 points) Suppose n and m are integers. Prove that, if $n + m > 10$, then $n > 5$ or $m > 5$.

Proof: Suppose n and m are integers and that $n \leq 5$ and $m \leq 5$.
Then, by EPI #12, $n + m \leq 10$.
So, if $n + m > 10$, then $n > 5$ or $m > 5$. □

3. (12 points) Suppose a, b, and c are integers and let R be the statement

R: $c \mid a$ or $c \mid b$ only if $c \mid ab$.

   (a) Rewrite R in "*if-then*" form and give its converse.
   R in "*if-then*" form: If $c \mid a$ or $c \mid b$, then $c \mid ab$.

   converse of R: If $c \mid ab$, then $c \mid a$ or $c \mid b$.

   (b) Is R true? If so, prove it. If not, give a counter-example that demonstrates that R is false.
   Yes!

   Proof: Suppose $c \mid a$ and $c \mid b$.
   Then there exist integers k and ℓ such that $a = ck$ and $b = c\ell$.
   Then $ab = (ck)(c\ell) = c(ck\ell)$.
   Thus, $c \mid ab$. □

   (c) Is the converse of R true? If so, prove it. If not, give a counter-example that demonstrates that the converse of R is false.
   No! Let $a = 2$, $b = 2$, and $c = 4$. Then $c \mid ab$ but $c \nmid a$ and $c \nmid b$.

4. (10 points) Suppose a and b are integers. Prove that $a^2 + 3b + 4$ is even if and only if a and b have the same parity.

Proof: Suppose a and b are integers.

$\Rightarrow$) Suppose, for the sake of contraposition, that a and b have opposite parity.

Case 1: Suppose a is even and b is odd.
Then $a = 2k$ and $b = 2\ell + 1$ for some $k, \ell \in \mathbb{Z}$.
Then $a^2 + 3b + 4 = (2k)^2 + 3(2\ell + 1) + 4 = 4k^2 + 6\ell + 7 = 2(2k^2 + 3\ell + 3) + 1$, which is odd.

Case 2: Suppose a is odd and b is even.
Then $a = 2k + 1$ and $b = 2\ell$ for some $k, \ell \in \mathbb{Z}$.
Then $a^2 + 3b + 4 = (2k + 1)^2 + 3(2\ell) + 4 = 4k^2 + 4k + 6\ell + 5 = 2(2k^2 + 2k + 3 + 2) + 1$, which is odd.

Thus, if $a^2 + 3b + 4$ is even, then a and b must have the same parity.

$\Leftarrow$) Conversely, suppose a and b have the same parity.

Case 1: Suppose a and b are both even.
Then $a = 2k$ and $b = 2\ell$ for some $k, \ell \in \mathbb{Z}$.
Then $a^2 + 3b + 4 = (2k)^2 + 3(2\ell) + 4 = 4k^2 + 6\ell + 4 = 2(2k^2 + 3\ell + 2)$, which is even.

Case 2: Suppose a and b are both odd.
Then $a = 2k + 1$ and $b = 2\ell + 1$ for some $k, \ell \in \mathbb{Z}$.
Then $a^2 + 3b + 4 = (2k + 1)^2 + 3(2\ell + 1) + 4 = 4k^2 + 4k + 6\ell + 8 = 2(2k^2 + 2k + 3\ell + 4)$, which is even.

Thus, if a and b have the same parity, then $a^2 + 3b + 4$ is even. □

5. (8 points) Prove that, for all $n \in \mathbb{Z}$, there exists a unique $m \in \mathbb{Z}$ such that $5n - 2m = 3m + 10$.

Proof: Suppose $n \in \mathbb{Z}$.

Let $m = n - 2$.

Then $m \in \mathbb{Z}$.

Further, $5n - 2m = 5n - 2(n - 2) = 3n + 4$ and $3m + 10 = 3(n - 2) + 10 = 3n - 6 + 10 = 3n + 4$.

Thus, $5n - 2m = 3m + 10$.

To show that this m is unique, suppose k is an integer with the property that $5n - 2k = 3k + 10$.

Then $5k + 10 = 5n$.

Similarly, since $5n - 2m = 3m + 10$, $5n = 5m + 10$.

This implies that $5k + 10 = 5m + 10$ which in turn implies that $5k = 5m$.

We then have that $5(k - m) = 0$ and, since $5 \neq 0$, this means $k - m = 0$ and thus $k = m$.

Therefore, for m is the unique integer such that $5n - 2m = 3m + 10$. □