Commentary on a needle experiment - Matthew M. Conroy

We imagine a needle experiment in which the plane is crossed by two orthogonal sets of parallel lines, which cut the plane into squares. Further, each square is then cut by a diagaonal, and we imagine dropping needles to estimate the probability that the needle crosses some line (diagonal, or square boundary).

We consider the case in which the needle has length equal to the side-length of the squares.

We could write a simulation to estimate the probability.

Alternatively, we can try to work this probability out exactly.

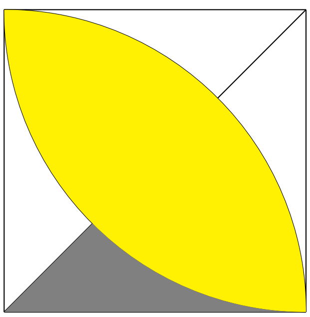
The approach is to define a function $p(x, y)$ that gives the probability that, if one end of the needle lands at the point (x, y), the needle crosses the diagonal or square edge.

Then the probability we seek is

$$P = \iint_S p(x, y)\, dA = \int_0^1 \int_0^1 p(x, y)\, dx dy$$

where S is the square $[0, 1] \times [0, 1]$.

If we draw circular arcs with radius one from opposite corners, we have the following image.



Notice that if one end of the needle falls in the yellow region, then the needle will definitely cross the diagonal or a square edge.

The non-yellow part of the square can be cut into four identical regions like the one shaded grey.

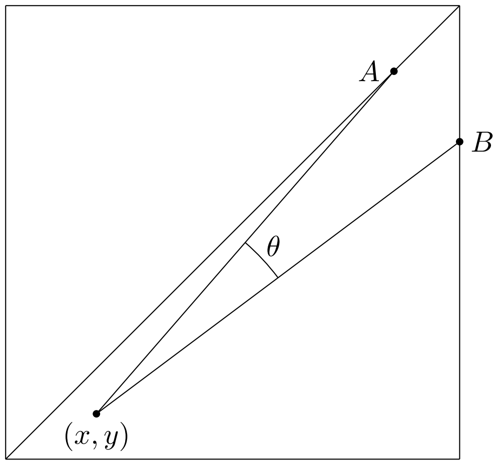
Hence,

$$P = 1 - 4 \iint_G (1 - p(x, y))\, dA$$

where G is the grey-shaded region.

Suppose one end of the needle lands at the point (x, y) in the grey region.

Then, in order for the needle to *not* cross the diagonal or a square edge, the other end of the needle must fall "between" points A and B as in the diagram below.



Hence, the probability that the needle does not cross the diagonal or a square edge is

$$1 - p(x, y) = \frac{\theta}{2\pi}$$

where

$$\theta = t(x, y) = \cos^{-1}\left(\frac{x + y + \sqrt{2 - (x-y)^2}}{2} - x\right) - \cos^{-1}(1 - x)$$

Thus,

$$\begin{aligned} P &= 1 - 4 \iint\limits_{G} \frac{t(x, y)}{2\pi}\, dA \\ &= 1 - 4\left(\int_0^{1-\frac{1}{\sqrt{2}}} \int_0^x \frac{t(x, y)}{2\pi}\, dy\, dx + \int_{1-\frac{1}{\sqrt{2}}}^1 \int_0^{1-\sqrt{1-(x-1)^2}} \frac{t(x, y)}{2\pi}\, dy\, dx\right) \end{aligned}$$

Using a numerical integrator, we can find that

$$P = 0.99116137306521....$$