Here are proofs of two theorems in Homework 4.

I have added comments in brackets.

Theorem Let A and B be sets. Then $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$, with equality if and only if

$$A \subseteq B \text{ or } B \subseteq A.$$

Proof Let A and B be sets.

[We begin by proving that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$ completely generally.]

Suppose $x \in \mathcal{P}(A) \cup \mathcal{P}(B)$.

Then $x \in \mathcal{P}(A)$ or $x \in \mathcal{P}(B)$.

Hence, $x \subseteq A$ or $x \subseteq B$.

[We need to show that $x \in \mathcal{P}(A \cup B)$. That is, we need to show that $x \subseteq A \cup B$.]

Suppose $y \in x$.

Then $y \in A$ or $y \in B$ (since $x \subseteq A$ or $x \subseteq B$).

Hence, $y \in A \cup B$.

Thus, $y \in x$ implies $y \in A \cup B$, and so $x \subseteq A \cup B$.

Hence, $x \in P(A \cup B)$.

Therefore, $x \in \mathcal{P}(A) \cup \mathcal{P}(B)$ implies $x \in P(A \cup B)$, and so

$$\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B).$$

Now, we wish to show that $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$ if and only if $A \subseteq B$ or $B \subseteq A$.

Since we just showed $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$, we need to show that

$$\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$$

if and only if $A \subseteq B$ or $B \subseteq A$.

[Since this is an "if and only if" statement, we have two directions to prove.]

We start with "reverse" direction.

Suppose $A \subseteq B$ or $B \subseteq A$.

Without loss of generality, let's assume $A \subseteq B$.

Suppose $x \in \mathcal{P}(A \cup B)$.

Then $x \subseteq A \cup B$.

Suppose $y \in x$.

Then $y \in A$ or $y \in B$.

Since $A \subseteq B$, $y \in A$ implies $y \in B$.

Hence, $y \in B$.

So, $y \in x$ implies $y \in B$, and hence $x \subseteq B$.

That is, $x \in \mathcal{P}(B)$, and so $x \in \mathcal{P}(A) \cup \mathcal{P}(B)$.

Thus, we have shown that $x \in \mathcal{P}(A \cup B)$ implies $x \in \mathcal{P}(A) \cup \mathcal{P}(B)$, and so

$$\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B).$$

Now, we prove the "forward" direction by proving the contrapositive

Suppose $A \nsubseteq B$ and $B \nsubseteq A$.

Then there is an $x \in A$ such that $x \notin B$, and there is a $y \in B$ such that $y \notin A$.

Let $S = \{x, y\}$.

Suppose $z \in S$.

Then $z = x$ or $z = y$, so $z \in A$ or $z \in B$; that is, $z \in A \cup B$.

Hence, $S \subseteq A \cup B$; that is, $S \in \mathcal{P}(A \cup B)$.

On the other hand, $S \nsubseteq A$ since $y \in S$ and $y \notin A$.

Also, $S \nsubseteq B$ since $x \in S$ and $x \notin B$.

Hence, $S \notin \mathcal{P}(A)$ and $S \notin \mathcal{P}(B)$, and so $S \notin \mathcal{P}(A) \cup \mathcal{P}(B)$.

Thus $\mathcal{P}(A \cup B) \nsubseteq \mathcal{P}(A) \cup \mathcal{P}(B)$.

Hence, $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$ implies $A \subseteq B$ or $B \subseteq A$.

Thus, $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$ if and only if $A \subseteq B$ or $B \subseteq A$. ■

Theorem Suppose $\mathcal{R}$ and $\mathcal{S}$ are families of sets. Then $(\cup\mathcal{R}) \setminus (\cup\mathcal{S}) \subseteq \cup(\mathcal{R} \setminus \mathcal{S})$.

Proof Suppose $\mathcal{R}$ and $\mathcal{S}$ are families of sets.

Suppose $x \in (\cup\mathcal{R}) \setminus (\cup\mathcal{S})$.

Then $x \in \cup\mathcal{R}$ and $x \notin \cup\mathcal{S}$.

Since $x \in \cup\mathcal{R}$, there exists a set $A \in \mathcal{R}$ such that $x \in A$.

[At this point, we need to argue that $x \in \cup(\mathcal{R} \setminus \mathcal{S})$. To do that, we need to find a set in $(\mathcal{R} \setminus \mathcal{S})$ that x is in. A is that set, as we now show.]

Suppose $A \in \mathcal{S}$.

Then $x \in \cup\mathcal{S}$.

This is a contradiction, since we know $x \notin \mathcal{S}$, and so $A \notin \mathcal{S}$.

Thus $A \in \mathcal{R}$ and $A \notin \mathcal{S}$, and so $A \in \mathcal{R} \setminus \mathcal{S}$.

Since $x \in A$, $x \in \cup(\mathcal{R} \setminus \mathcal{S})$.

Thus $x \in (\cup\mathcal{R}) \setminus (\cup\mathcal{S})$ implies $x \in \cup(\mathcal{R} \setminus \mathcal{S})$.

Hence, $(\cup\mathcal{R}) \setminus (\cup\mathcal{S}) \subseteq \cup(\mathcal{R} \setminus \mathcal{S})$. ■