MATH 300 — Dr. Matthew Conroy
Final Exam Practice Problems

NOTE: For this set of problems, we will use the convention that $\mathbb{N}$ is the set of positive integers, i.e. $\mathbb{N} = \{1, 2, 3, 4, 5, ...\}$ and does not include zero.

1. For each of the following, determine whether the statement is TRUE or FALSE. You do not need to provide any justification of your answer.
   (a) Every infinite subset of $\mathbb{R}$ is uncountable.
   (b) There is an uncountable subset of $\mathbb{N} \times \mathbb{N}$.
   (c) There exists a bijection $f : \mathbb{Q} \to \mathbb{R}$.
   (d) There exists a bijection $f : \mathbb{Q} \to \mathbb{Z}$.
   (e) If $f : \mathbb{R} \to \mathbb{R}$ is surjective, then f must be bijective.
   (f) If $f : \mathbb{R} \to \mathbb{R}$ is injective, then f must be surjective.
   (g) Let A be a finite set. If $f : A \to A$ is surjective, then f must be injective.
   (h) Let A be a finite set. If $f : A \to A$ is injective, then f must be bijective.
   (i) Suppose the relation R on $\mathbb{Z}$ is defined by $(a, b) \in R \Leftrightarrow a < b$. R is an equivalence relation on $\mathbb{Z}$.
   (j) If a, b, and c are integers, $c \neq 0$, and $c|ab$, then it must be the case that $c|a$ or $c|b$.

2. Use induction to show that, if x is a real number such that $1 + x > 0$, then $(1 + x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.

3. We proved in class that
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \text{ for all } n \in \mathbb{N}.$$
   Use this fact and induction to prove that $\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$ for all $n \in \mathbb{N}$.

4. Prove that $6|(n^3 - n)$ for every $n \in \mathbb{N}$.

5. Prove that $3|(7^n - 4)$ for every $n \in \mathbb{N}$.

6. Define a relation T on the set $\mathbb{R}$ of real numbers by
$$T = \{(x, y) \in \mathbb{R} \times \mathbb{R} : |x - y| < 1\}.$$
   Is T an equivalence relation? (Justify your answer, of course.)

7. (a) Define a relation R on $\mathbb{N}$ by
$$(x, y) \in R \Leftrightarrow x - y \text{ is even.}$$
   Prove that R is an equivalence relation.
   (b) Define a relation R on $\mathbb{Z}$ by
$$(x, y) \in R \Leftrightarrow xy \equiv 0 \pmod 4.$$
   Give a counterexample to demonstrate that R is not transitive.

8. Let A, B, and C be sets and consider functions $f : A \to B$ and $g : B \to C$. State whether each of the following is true or false. If the statement is true, prove it; if it is false, give a counterexample.

   (a) If $g \circ f : A \to C$ is injective, then f must be injective.

   (b) If $g \circ f : A \to C$ is surjective, then f must be surjective.

9. Let $A = \{x \in \mathbb{R} : x \neq 1\}$ and define $f : A \to \mathbb{R}$ by

$$f(x) = \frac{x+1}{x-1}.$$

   Is $f(x)$ injective? surjective? bijective? Justify each of your responses with a proof or counterexample.

10. Define a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ by $f(x, y) = (x - y, 2x + y)$. Is f one-to-one? onto? Justify each of your responses with a proof or counterexample.

11. Let $A = \{a \in \mathbb{N} : a \text{ is even}\}$ and $B = \{b \in \mathbb{N} : b \text{ is odd}\}$.

    (a) Define a function $f : A \times B \to \mathbb{N}$ by $f(a, b) = \frac{ab}{2}$. Is f surjective? Justify your answer.

    (b) Define a function $h : A \times B \to \mathbb{N}$ by $h(a, b) = \frac{a + 2b}{2}$. Is h surjective? Justify your answer.

    (c) Define a function $g : B \to \mathbb{N}$ by $g(b) = \frac{b+1}{2}$. Prove that g is bijective.

12. Let $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \neq 2\}$ and define $f : S \to S$ by

$$f(x, y) = \left(\frac{y+2}{x-2}, \frac{1}{x-2} \right).$$

    (a) Prove that f is injective.

    (b) Is f bijective? Prove it or give a counterexample.

13. Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. Define $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = f(x) + g(x)$. For each of the following, if the statement is true, prove it; otherwise, give a counterexample to show that the statement is false.

    (a) If f and g are bijections, then h is a bijection.

    (b) If f and g are both increasing, then h is increasing. (A function is an increasing function if $x_1 > x_2$ implies $f(x_1) > f(x_2)$.)

    (c) If f is increasing and g is decreasing, then $g \circ f$ is decreasing.

14. Let A be a set, and suppose there exists a function $f : \mathcal{P}(A) \to \mathbb{Z}$ which is a bijection. Prove that A is countable.

15. Suppose A and B are sets. Suppose A is finite. Prove that $A \sim B$ if and only if B is finite and $|A| = |B|$.