Below are some extra examples for Math 300. These examples are to augment the lectures and help you do some of the assigned homework problems.

1. To show $(P \wedge R) \vee (\neg R \wedge (P \vee Q))$ is equivalent to $P \vee (\neg R \wedge Q)$:

$$(P \wedge R) \vee (\neg R \wedge (P \vee Q))$$
is equivalent to
$$(P \wedge R) \vee (\neg R \wedge P) \vee (\neg R \wedge Q) \quad \text{(distributive and associative laws)}$$
is equivalent to
$$((P \wedge R) \vee \neg R) \wedge ((P \wedge R) \vee P)) \vee (\neg R \wedge Q) \quad \text{(distributive law)}$$
is equivalent to
$$(((P \wedge R) \vee \neg R) \wedge P) \vee (\neg R \wedge Q) \quad \text{(absorbtion)}$$
is equivalent to
$$((\neg R \vee P) \wedge (\neg R \vee R) \wedge P) \vee (\neg R \wedge Q) \quad \text{(distributive law)}$$
is equivalent to
$$P \vee (\neg R \wedge Q) \quad \text{(absorbtion and tautology)}$$

2. Enumeration of all logical connective possibilities.

	P Q F F	P Q F T	P Q T F	P Q T T
contradiction	F	F	F	F
$P \wedge Q$	F	F	F	T
$P \wedge \neg Q$	F	F	T	F
P	F	F	T	T
$\neg P \wedge Q$	F	T	F	F
Q	F	T	F	T
$P \oplus Q$	F	T	T	F
$P \vee Q$	F	T	T	T
$\neg(P \vee Q)$	T	F	F	F
$P \leftrightarrow Q$, $\neg(P \oplus Q)$	T	F	F	T
$\neg Q$	T	F	T	F
$Q \to P, P \vee \neg Q$	T	F	T	T
$\neg P$	T	T	F	F
$P \to Q, \neg P \vee Q$	T	T	F	T
$\neg(P \wedge Q)$	T	T	T	F
tautology	T	T	T	T

3. After working with the binary connectives, $\vee$ and $\wedge$, we might wonder if there are ternary connectives. First, some notation. We could define new notation for the binary connectives. For instance, we could define $(A, B)_1$ to be equivalent to $A \wedge B$ and $(A, B)_2$ to be equivalent to $A \vee B$.

   From the table above, we could define 16 different such connectives, i.e., $(A, B)_1, (A, B)_2, ..., (A, B)_16$.

   Extending this notation, we could define a ternary connective $(A, B, C)_1$ with the following truth table:

A	B	C	$(A, B, C)_1$
F	F	F	T
F	F	T	F
F	T	F	F
F	T	T	F
T	F	F	T
T	F	T	T
T	T	F	F
T	T	T	T

I just picked F and T it the right-most column at random.

Now, here's the interesting part: we can show this is equivalent to an expression using only A, B, C, $\vee$, $\wedge$ and $\neg$. Here's how.

Consider just the rows that have T in the right-most column. For each such row, consider the expression

$$(\neg)A \wedge (\neg)B \wedge (\neg)C$$

where the $\neg$ if there is there is a F in that statements column, and no $\neg$ otherwise.

For instance, for the first row, we have

$$\neg A \wedge \neg B \wedge \neg C$$

and for the fifth row, we have

$$A \wedge \neg B \wedge \neg C$$

We can create these expressions for each needed row. If we string these expressions together with $\vee$, we will have an expression that is equivalent to $(A, B, C)_1$:

$$(\neg A \wedge \neg B \wedge \neg C) \vee (A \wedge \neg B \wedge \neg C) \vee (A \wedge B \wedge C) \vee (A \wedge B \wedge C)$$

(There is certainly a chance that this could be simplified, of course.)

This process shows that *any* ternary connective can be expressed using only $\vee$, $\wedge$ and $\neg$. In fact, we can extend this to any number of input statements: more columns on the left in the table would not have effected our procedure at all. So even though we can imagine complex many-input logical constructs, they are all equivalent to expressible with just $\vee$, $\wedge$ and $\neg$.

(And as we'll see elsewhere, you can get by with just $\vee$ and $\neg$, or just $\wedge$ and $\neg$.)

4. Suppose we wish to show that $\exists x(P(x) \rightarrow Q(x))$ is equivalent to $\forall x P(x) \rightarrow \exists x Q(x)$.

   By the conditional laws (given on page 47 of Velleman),

$$\exists x(P(x) \rightarrow Q(x))$$

   is equivalent to

$$\exists x(\neg P(x) \vee Q(x)). \tag{1}$$

   Now, any statement $\exists x(R(x) \vee S(x))$ means that there exists an x such that $R(x)$ or $S(x)$ is true. That is equivalent to saying that there exists an x such that $R(x)$ is true, or there exists an x such that $S(x)$ is true. That is,

$$\exists x(R(x) \vee S(x)) \text{ is equivalent to } \exists x R(x) \vee \exists x S(x)$$

Thus, expression (1) is equivalent to

$$\exists x \neg P(x) \vee \exists x Q(x)$$

and this is equivalent, by our quantifier negation laws, to

$$\neg \forall x P(x) \vee \exists x Q(x)$$

and this is equivalent to

$$\forall x P(x) \rightarrow \exists x Q(x)$$

by our conditional laws.

Along these same lines, note that

$$\forall x (R(x) \wedge S(x))$$

says that for all x, R(x) and S(x) are true. This is equivalent to saying that for all x R(x) is true and for all x S(x) is true (the latter just takes longer to say!). Hence, $\forall x (R(x) \wedge S(x))$ is equivalent to

$$\forall x R(x) \wedge \forall x S(x).$$

5. Why can we prove by cases?

   Suppose we want to show $A \rightarrow S$.

   Suppose we know B or C is true (e.g., for an integer n, we might set $B =$"n is even" and $C =$"n is odd").

   Then we use:

$$\begin{aligned} &((A \wedge B) \rightarrow S) \wedge ((A \wedge C) \rightarrow S) \\ \Leftrightarrow\ &(\neg(A \wedge B) \vee S) \wedge (\neg(A \wedge C) \vee S) \\ \Leftrightarrow\ &(\neg A \vee \neg B \vee S) \wedge (\neg A \vee \neg C \vee S) \\ \Leftrightarrow\ &((\neg A \vee \neg B) \wedge (\neg A \vee \neg C)) \vee S \\ \Leftrightarrow\ &(\neg A \vee (\neg B \wedge \neg C)) \vee S \\ \Leftrightarrow\ &\neg A \vee S \\ \Leftrightarrow\ &A \rightarrow S. \end{aligned}$$

   Note we use the fact that $\neg B \wedge \neg C$ is false here.

6. An example of a uniqueness argument.

   Theorem: Let S be a set. For every $A \in \mathcal{P}(S)$, there is a unique $B \in \mathcal{P}(S)$ such that for every $C \in \mathcal{P}(S)$, $C \setminus A = C \cap B$.

   Proof. Let S be a set. Let $A \in \mathcal{P}(S)$. Let $B = S \setminus A$. Then, let $C \in \mathcal{P}(S)$.

   Suppose $x \in C \setminus A$. So, $x \in C$ and $x \notin A$, so $x \in B$.

   Hence, $x \in B \cap C$. Thus, $C \setminus A \subseteq C \cap B$.

   Suppose $x \in C \cap B$. Then $x \in C$ and $x \in B$, so $x \notin A$.

   Hence, $x \in C \setminus A$. Thus, $C \cap B \subseteq C \setminus A$.

   Thus, $C \cap B = C \setminus A$.

Thus there exists a set with the required property, namely $B = S \setminus A$.

To show uniqueness, suppose a set D in $\mathcal{P}(S)$ also has the required property.

Then for all $C \in \mathcal{P}(S), C \cap D = C \setminus A$.

Since for all $C \in \mathcal{P}(S), C \cap B = C \setminus A$, we have that for all $C \in \mathcal{P}(S)$,

$$C \cap B = C \cap D.$$

In particular, if we let C=B, we have $B \cap B = B \cap D$, i.e., $B = B \cap D$. This shows that $B \subseteq D$, since if $x \in B$, x is also in D.

On the other hand, if we let C=D, we have $D \cap D = D \cap B$, i.e., $D = D \cap B$. This shows that $D \subseteq B$, since if $x \in D$, x is also in B.

Hence, $D = B$. And so the choice of B is unique. □

7. A surprising bijection.

Let's consider the intervals of real numbers $A = (0, 1]$ and $B = (0, 1)$. Since both A and B contain an infinite number of elements, and B is simply A with one element (1) removed, it would be surprising if A and B were not equinumerous. However, if we try to find a bijection from one set to the other using combinations of our standard functions (e.g. trig functions, rational functions, etc.), we find that a bijection eludes us.

It turns out that it is possible, but it requires a surprising bit of cleverness. First, consider this function from A to B:

$$f(x) = \begin{cases} 1/2 & \text{if } x = 1, \\ x & \text{if } x \neq 1. \end{cases}$$

Notice that this *almost* works. It maps the set $(0, 1]$ onto the set $(0, 1)$, but it is not one-to-one: $f(1) = f(1/2)$.

To fix this, we might try sending 1/2 to something else:

$$g(x) = \begin{cases} 1/3 & \text{if } x = 1/2, \\ 1/2 & \text{if } x = 1, \\ x & \text{if } x \neq 1 \text{ and } x \neq 1/2. \end{cases}$$

Again, g maps the set $(0, 1]$ onto $(0, 1)$ and now $f(1) \neq f(1/2)$, but now $f(1/2) = f(1/3)$ so it is again not one-to-one.

One more try: let's send 1/3 to something else:

$$h(x) = \begin{cases} 1/4 & \text{if } x = 1/3, \\ 1/3 & \text{if } x = 1/2, \\ 1/2 & \text{if } x = 1, \\ x & \text{if } x \neq 1, x \neq 1/2, \text{ and } x \neq 1/3. \end{cases}$$

This function maps the set $(0, 1]$ onto $(0, 1)$ and $h(1) \neq h(1/2)$ *and* $h(1/2) \neq h(1/3)$, but now $h(1/3) = h(1/4)$, so h is not one-to-one.

Although this doesn't seem to be working, if we *extend this strategy forever*, we get a function that does work.

Let $k : A \to B$ be defined like this:

$$k(x) = \begin{cases} \frac{1}{n+1} & \text{if } x = 1/n \text{ for some } n \in \mathbb{Z}_{>0}, \\ x & \text{if } x \neq 1/n \text{ for all } n \in \mathbb{Z}_{>0}. \end{cases}$$

So we have $k(1) = 1/2$, $k(1/2) = 1/3$, $k(1/3) = 1/4$, $k(1/4) = 1/5$, etc. We are basically pushing our problem down the sequence of $1/n$, and since this sequence is infinite, it becomes a non-problem.

It is not too hard to show that our k function is a bijection from A to B, and this shows that A and B are equinumerous.

8. An *algebraic number* is a number that is the root of a polynomial with rational coefficients. One can prove that if x is an algebraic number, then x^2 is an algebraic number. Here is the proof.

   Suppose $\hat{x}$ is algebraic.

   Then $P(\hat{x}) = 0$ for some polynomial P with rational coefficients. Let's write P as a sum of even exponent terms and odd exponent terms. That is, write $P(x) = A(x) + B(x)$ where all the exponents in A are even and all the exponents in B are odd.

   For example, if $P(x) = 2x^5 - 3x^4 - \frac{1}{2}x^3 + 7x^2 + x + 5$, then $P(x) = (-3x^4 + 7x^2 + 5) + (2x^5 - \frac{1}{2}x^3 + x)$, so we could take $A(x) = -3x^4 + 7x^2 + 5$ and $B(x) = 2x^5 - \frac{1}{2}x^3 + x$.

   Now, A can be viewed as a polynomial in x^2, and B can be viewed as x times a polynomial with only even exponents, so we have $P(x) = A'(x^2) + xB'(x^2)$, where A' and B' are polynomials with rational coefficients. Then

   $$P(\hat{x}) = A'(\hat{x}^2) + \hat{x}B'(\hat{x}^2) = 0.$$

   Then

   $$(A'(\hat{x}^2) + \hat{x}B'(\hat{x}^2))(A'(\hat{x}^2) - \hat{x}B'(\hat{x}^2)) = (A'(\hat{x}^2))^2 - \hat{x}^2(B'(\hat{x}^2))^2 = 0.$$

   That is, $\hat{x}^2$ is a root of

   $$(A'(x^2))^2 - x^2(B'(x^2))^2$$

   and, since $(A')^2$ and $(B')^2$ have rational coefficients, we can conclude that $\hat{x}^2$ is algebraic. ■