MATH 300 C - Spring 2015
Midterm 2 Practice Problems

1. *Prove that, for all integers n, 3 does not divide $n^2 - 5$.*

   Let n be an integer.

   By Euclid's theorem, $n = 3k + r$ for some integers k and r, and $0 \leq r < 3$.

   That is, $r = 0, 1,$ or 2.

   Then $n^2 - 5 = (3k + r)^2 - 5 = 9k^2 + 6kr + r^2 - 5 = 9k^2 + 6kr + r^2 - 6 + 1 = 3(3k^2 + 2kr - 2) + r^2 + 1$.

   Let t be the remainder when $n^2 - 5$ is divided by 3.

   Then t is equal to the remainder then $r^2 + 1$ is divided by 3.

   If $r = 0$, then $r^2 + 1 = 1 = 0 \cdot 3 + 1$, so $t = 1$.

   If $r = 1$, then $r^2 + 1 = 2 = 0 \cdot 3 + 2$, so $t = 2$.

   If $r = 2$, then $r^2 + 1 = 5 = 1 \cdot 3 + 2$, so $t = 2$.

   Hence the remainder when $n^2 - 5$ is divided by 3 is 1 or 2 and not zero.

   Thus 3 does not divide $n^2 - 5$. ■

2. *Define a relation R on $\mathbb{Z}$ by*
   $$(x, y) \in R \Leftrightarrow 4 \mid x^2 - y^2.$$
   Is R an equivalence relation? Prove your answer.

   Let $x \in \mathbb{Z}$.

   Then $x^2 - x^2 = 0 = 4 \cdot 0$, so $4 \mid x^2 - x^2$.

   Hence, $(x, x) \in R$, and so R is reflexive.

   Suppose $(x, y) \in R$.

   Then $4 \mid x^2 - y^2$.

   That is, $x^2 - y^2 = 4k$ for some integer k.

   Then $y^2 - x^2 = -4k = 4(-k)$ and since $-k \in \mathbb{Z}$, $4 \mid y^2 - x^2$.

   Hence, $(y, x) \in R$, and so R is symmetric.

   Suppose $(x, y) \in R$ and $(y, z) \in R$.

   Then $4 \mid x^2 - y^2$, so $x^2 - y^2 = 4k$ for some integer k.

   Also, $4 \mid y^2 - z^2$, so $y^2 - z^2 = 4m$ for some integer m.

   Hence, $x^2 - y^2 + y^2 - z^2 = x^2 - z^2 = 4k + 4m = 4(k + m)$.

   Since $k + m$ is an integer, $4 \mid x^2 - z^2$, and so $(x, z) \in R$.

   Thus, R is transitive.

   Since R is reflexive, symmetric and transitive, R is an equivalence relation. ■

3. *Use induction to prove that*

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

for all $n \in \mathbb{Z}_{>0}$.

Let $n \in \mathbb{Z}_{>0}$.

Let $P(n)$ be the statement "$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$".

Base Case: Let $n = 1$.

Then

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1(1+1)} = \frac{1}{2} = \frac{1}{1+1} = \frac{n}{n+1}.$$

Thus, $P(1)$ is true.

Induction Step: Suppose $P(n)$ is true for some $n = k > 0$.

So $P(k)$ is true. That is,

$$\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$$

Then

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{i(i+1)} &= \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k(k+2)+1}{(k+1)(k+2)} \\
&= \frac{k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{(k+1)^2}{(k+1)(k+2))} \\
&= \frac{k+1}{k+2}
\end{aligned}$$

Hence $P(k+1)$ is true.

Thus, $P(k)$ implies $P(k+1)$, and $P(1)$ is true, so, by induction, $P(n)$ is true for all $n \in \mathbb{Z}_{>0}$.■

4. *Suppose* A, B *and* C *are sets. Suppose* $f : A \to B$ *and* $g : B \to C$.

(a) *Prove that if f is onto and g is not one-to-one, then $g \circ f$ is not one-to-one.*
Suppose A, B and C are sets. Suppose $f : A \to B$ and $g : B \to C$, f is onto and g is not one-to-one.
Since g is not one-to-one, there exist $b_1 \neq b_2 \in B$ such that $g(b_1) = g(b_2)$.
Since f is onto, there exist a_1 and a_2 such that $f(a_1) = b_1$ and $f(a_2) = b_2$.
Note that $a_1 \neq a_2$ since $b_1 \neq b_2$.
Then $g(f(a_1)) = g(b_1) = g(b_2) = g(f(a_2))$, so $g \circ f$ is not one-to-one.■

(b) *Prove that if f is not onto and g is one-to-one, then $g \circ f$ is not onto.*
Suppose A, B and C are sets. Suppose $f : A \to B$ and $g : B \to C$, f is not onto and g is one-to-one.
Since f is not onto, there is a $b \in B$ such that $f(a) \neq b$ for all $a \in A$.
Suppose $g(f(a)) = g(b)$ for some $a \in A$.
Then, since g is one-to-one, $f(a) = b$.
This is a contradiction: for all $a \in A$, $f(a) \neq b$.
Thus, for all $a \in A$, $g(f(a)) \neq g(b)$.
Hence, $g \circ f$ is not onto. ■

5. *Let $A = \mathbb{R} \times \mathbb{R} \setminus \{(0, 0)\}$.*
Thus, A is the xy-plane without the origin.
Define a relation R on A by

$$((x_1, y_1), (x_2, y_2)) \in R \Leftrightarrow (x_1, y_1) \text{ and } (x_2, y_2) \text{ lie on a line which passes through the origin.}$$

Prove that R is an equivalence relation.

Is R reflexive?

Suppose $P \in A$. Then $P = (x_1, y_1)$ for some $x_1, y_1 \in \mathbb{R}$.

Suppose $x_1 = 0$. Then P lies on the line $x = 0$ which passes through the origin.

Hence, $(P, P) \in R$.

Suppose $x_1 \neq 0$. Then P lies on the line $y = \frac{y_1}{x_1}x$, which passes through the origin.

Hence, $(P, P) \in R$.

Thus, in all cases, $(P, P) \in R$, so R is reflexive.

Is R symmetric?

Suppose $P = (x_1, y_1) \in A$ and $Q = (x_2, y_2) \in A$ and $(P, Q) \in R$.

Then P and Q lie on a line through the origin, and so Q and P lie on a line through the origin.

Hence, $(Q, P) \in R$, and so R is symmetric.

Is R transitive?

Suppose $P = (x_1, y_1) \in A$ and $Q = (x_2, y_2) \in A$ and $S = (x_3, y_3) \in A$ and $(P, Q) \in R$ and $(Q, S) \in R$.

Suppose $x_1 = 0$.

Then P lies on the vertical line $x = 0$ through the origin and no other line through the origin.

Hence, Q lies on $x = 0$, and hence S lies on $x = 0$.

Thus, P and S lie on a line through the origin, and so $(P, S) \in R$.

Suppose $x_1 \neq 0$.

Then P and Q lie on a line $y = mx$ where $m \in \mathbb{R}$, and Q and R lie on a line $y = nx$ where $n \in \mathbb{R}$.

Then $y_2 = mx_2 = nx_2$ so $(m - n)x_2 = 0$.

If $x_2 = 0$, then $y_2 = mx_2 = 0$, so $Q = (0, 0) \notin A$, a contradiction since $Q \in A$.

Hence $x_2 \neq 0$, so $m - n = 0$, i.e., $m = n$.

Thus, P and Q lie on the line $y = mx$ and Q and S lie on the line $y = mx$, so P and S both lie on a line through the origin.

Hence, $(P, S)\ inR$ and thus R is transitive.

Since R is reflexive, symmetric, and transitive, R is an equivalence relation. ■

6. *Let $S = \mathbb{R}_{>0} \times \mathbb{R}_{>0}$. Define a relation $R \subseteq S \times S$ by*

$$((x_1, y_1), (x_2, y_2)) \in R \Leftrightarrow x_1y_1 = x_2y_2.$$

Prove that R is an equivalence relation.

Suppose $P = (x, y) \in S \times S$.

Since $xy = xy$, $((x, y), (x, y) \in R$, i.e., $(P, P) \in R$.

Hence, R is reflexive.

Suppose $((x_1, y_1), (x_2, y_2)) \in R$.

Then $x_1y_1 = x_2y_2$.

Hence, $x_2y_2 = x_1y_1$, so $((x_2, y_2), (x_1, y_1)) \in R$.

Thus, R is symmetric.

Suppose $((x_1, y_1), (x_2, y_2)) \in R$ and $((x_2, y_2), (x_3, y_3)) \in R$.

Then $x_1y_1 = x_2y_2$ and $x_2y_2 = x_3y_3$.

Hence, $x_1y_1 = x_3y_3$.

Thus, $((x_1, y_1), (x_3, y_3)) \in R$.

Hence, R is transitive.

Since R is reflexive, symmetric and transitive, R is an equivalence relation.■

7. *Let $\mathcal{F}$ be a family of sets, and B be a set. Prove that if $\bigcup \mathcal{F} \subseteq B$, then $\mathcal{F} \subseteq \mathcal{P}(B)$.*

Let $\mathcal{F}$ be a family of sets, and B be a set.

Suppose $\bigcup \mathcal{F} \subseteq B$.

Let $x \in \mathcal{F}$.

Let $y \in x$.

Then $y \in \bigcup \mathcal{F}$.

Hence, $y \in B$.

Since $y \in x$ implies $y \in B$, $x \subseteq B$.

Hence, $x \in \mathcal{P}(B)$.

Since $x \in \mathcal{F}$ implies $x \in \mathcal{P}(B)$, $\mathcal{F} \subseteq \mathcal{P}(B)$. ■

8. *Let $\mathcal{F}$ and $\mathcal{G}$ be families of sets. Prove that $(\cap\mathcal{F}) \cap (\cap\mathcal{G}) = \cap(\mathcal{F} \cup \mathcal{G})$.*

Proof: Let $\mathcal{F}$ and $\mathcal{G}$ be families of sets.

Suppose $x \in \cap\mathcal{F} \cap \cap\mathcal{G}$.

Then $x \in \cap\mathcal{F}$ and $x \in \cap\mathcal{G}$.

Suppose $M \in \mathcal{F}$. Then $x \in M$.

Suppose $N \in \mathcal{G}$. Then $x \in N$.

Suppose $P \in \mathcal{F} \cup \mathcal{G}$. Then $P \in \mathcal{F}$ or $P \in \mathcal{G}$. Hence, $x \in P$.

Thus, x is an element of every set in $\mathcal{F} \cup \mathcal{G}$, so $x \in \cap(\mathcal{F} \cup \mathcal{G})$.

Hence, $(\cap\mathcal{F}) \cap (\cap\mathcal{G}) \subseteq \cap(\mathcal{F} \cup \mathcal{G})$.

Now, suppose $x \in \cap(\mathcal{F} \cup \mathcal{G})$.

Suppose $M \in \mathcal{F} \cup \mathcal{G}$. Then $x \in M$.

Suppose $N \in \mathcal{F}$. Then $N \in \mathcal{F} \cup \mathcal{G}$, and so $x \in N$.

Hence, x is in every set in $\mathcal{F}$, and so $x \in \cap\mathcal{F}$.

Suppose $P \in \mathcal{G}$. Then $P \in \mathcal{F} \cup \mathcal{G}$, and so $x \in P$.

Hence, x is in every set in $\mathcal{G}$, and so $x \in \cap\mathcal{G}$.

Thus, $x \in (\cap\mathcal{F}) \cap (\cap\mathcal{G})$.

Therefore $\cap(\mathcal{F} \cup \mathcal{G}) \subseteq (\cap\mathcal{F}) \cap (\cap\mathcal{G})$, and so $(\cap\mathcal{F}) \cap (\cap\mathcal{G}) = \cap(\mathcal{F} \cup \mathcal{G})$. ■

9. *Give an example of a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that f is one-to-one, but not onto (i.e., f is injective but not surjective). Prove that f is one-to-one and not onto.*

> Note: There are many examples you could give. If you are using this exam solution to study, I recommend creating a few, as different from each other as possible.

One example is $f(x) = x + 1$.

We note that since $x > 0$, $x + 1 > 1$ and so there is no a such that $f(a) = 1$. So f is not onto.

Suppose $f(x_1) = f(x_2)$. Then $x_1 + 1 = x_2 + 1$, and so $x_1 = x_2$. So f is one-to-one. ■

10. *Give an example of a function $g : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that g is onto, but not one-to-one (i.e., g is surjective, but not injective). Prove that g is onto and not one-to-one.*

> Note: There are many examples you could give. If you are using this exam solution to study, I recommend creating a few, as different from each other as possible.

Here is one example. Let

$$g(x) = \begin{cases} x & \text{if } x \text{ is odd,} \\ x/2 & \text{if } x \text{ is even.} \end{cases}$$

Let a be a positive integer.

Suppose a is odd. Then $g(a) = a$.

Suppose a is even. Then $2a \in \mathbb{Z}_{>0}$ and $g(2a) = a$.

Thus, g is onto.

On the other hand, $g(6) = g(3) = 3$, so g is not one-to-one. ■

11. *Use induction to prove that $n! > n^2$ for all integers $n \geq 4$.*

Proof: Define $P(n)$ to be the statement "$n! > n^2$".

Base case: Let $n = 4$. Then $n! = 24 > 16 = n^2$, so $P(4)$ is true.

Induction step: Suppose $P(n)$ is true for some $n = x \geq 4$.

Then $x! > x^2$, i.e. $\dfrac{x!}{x^2} > 1$.

Then

$$\begin{aligned}\frac{(x+1)!}{(x+1)^2} &= \frac{(x+1)x!}{(x+1)^2\frac{x^2}{x^2}} \\ &= \frac{(x+1)x^2}{(x+1)^2}\left(\frac{x!}{x^2}\right) \\ &= \frac{x^2}{x+1}\left(\frac{x!}{x^2}\right) \\ &= \left(x-1+\frac{1}{x+1}\right)\left(\frac{x!}{x^2}\right)\end{aligned}$$

We note that $x - 1 + \frac{1}{x+1} \geq 4 - 1 + 0 = 3 > 1$, and so, since $\frac{x!}{x^2} > 1$,

$$\frac{(x+1)!}{(x+1)^2} > 1$$

i.e., $(x+1)! > x^2$. Thus, $P(x+1)$ is true.

Thus, $P(x)$ implies $P(x+1)$, and since $P(4)$ is true, by induction $P(n)$ is true for all integers $n \geq 4$. ■

12. *Let R be the relation defined on the real numbers, $\mathbb{R}$, by*

$$(x, y) \in R \Leftrightarrow \textit{there exist positive integers } n \textit{ and } m \textit{ such that } x^n = y^m.$$

Prove that R is an equivalence relation.

Proof: Let $x \in \mathbb{R}$. Then $x^1 = x^1$, and so $(x, x) \in R$.

Hence, R is reflexive.

Suppose $(x, y) \in R$. Then $x^m = y^n$ for some integers m and n.

Then, $y^n = x^m$, and so $(y, x) \in R$.

Hence R is symmetric.

Suppose $(x, y) \in R$ and $(y, z) \in R$.

Then $x^m = y^n$ and $y^r = z^s$ for positive integers m, n, r, and s.

Then $x^{rm} = y^{rn}$ and $y^{rn} = z^{sn}$, so $x^{rm} = z^{sn}$.

Since rm and sn are positive integers, we conclude that $(x, z) \in R$.

Hence, R is transitive.

Since R is reflexive, symmetric and transitive, R is an equivalence relation. ■

13. *Let A, B and C be sets. Let $f : A \to B$, and $g : B \to C$.*

    (a) *Suppose $g \circ f : A \to C$ is one-to-one. Is f necessarily one-to-one? Prove your answer.*
    f is necessarily one-to-one.
    Proof: Suppose f is not one-to-one.
    Then there exist $a_1, a_2 \in A$, $a_1 \neq a_2$, with $f(a_1) = f(a_2)$.
    Then $g(f(a_1)) = g(f(a_2))$. But $a_1 \neq a_2$, so $g \circ f$ is not one-to-one. This is a contradiction.
    Hence f is one-to-one. ■

    (b) *Suppose $g \circ f : A \to C$ is one-to-one. Is g necessarily one-to-one? Prove your answer.*
    g is not necessarily one-to-one.
    Proof: We may defined $A = \{a\}$, $B = \{b_1, b_2\}$, and $C = \{c\}$. Then define $f = \{(a, b_2)\}$, $g = \{(b_1, c), (b_2, c)\}$.
    Then $g \circ f = \{(a, c)\}$, and $g \circ f$ is one-to-one though g is not.
    Alternatively, define $A = \mathbb{Z}_{>0}$, $B = \mathbb{Z}$, and $C = \mathbb{Z}$.
    Let $f(x) = x$ and $g(x) = |x|$. Then $(g \circ f)(x) = |x|$ is one-to-one from $\mathbb{Z}_{>0}$ to $\mathbb{Z}$, but g is not one-to-one from $\mathbb{Z}$ to $\mathbb{Z}$. ■

14. *Let S be a set.*

    Define a function $f : \mathcal{P}(S) \to \mathcal{P}(S)$ by $f(A) = S \setminus A$ for all $A \in \mathcal{P}(S)$.

    Prove that f is a bijection.

    Proof: Suppose A_1 and $A_2 \in \mathcal{P}(S)$ with $f(A_1) = f(A_2)$. Then

    $$S \setminus A_1 = S \setminus A_2.$$

    Suppose $x \in A_1$ but $x \notin A_2$. Then $x \in S$ (since $A_1 \subset S$), and $x \in S \setminus A_2$, but $x \notin S \setminus A_1$. This is a contradiction to the fact that $S \setminus A_1 = S \setminus A_2$. Hence if $x \in A_1$ then $x \in A_2$; i.e, $A_1 \subset A_2$.

    Suppose $x \in A_2$ but $x \notin A_1$. Then $x \in S$ (since $A_2 \subset S$), and $x \in S \setminus A_1$, but $x \notin S \setminus A_2$. This is a contradiction to the fact that $S \setminus A_2 = S \setminus A_1$. Hence if $x \in A_2$ then $x \in A_1$; i.e, $A_2 \subset A_1$.

    Thus $A_1 = A_2$, so f is one-to-one.

    Suppose $B \in \mathcal{P}(S)$.

    Let $Z = S \setminus B$.

    Then

    $$\begin{aligned} f(Z) &= S \setminus (S \setminus B) \\ &= \{x \in S : x \notin (S \setminus B)\} \\ &= \{x \in S : \neg(x \in S \setminus B)\} \\ &= \{x \in S : \neg(x \in S \text{ and } x \notin B)\} \\ &= \{x \in S : x \notin S \text{ or } x \in B\} \\ &= \{x \in B\} \\ &= B. \end{aligned}$$

Thus f is onto.

Hence f is one-to-one and onto, i.e., it is a bijection. ■

15. *Let S be the set of all functions $f : \mathbb{R} \Rightarrow \mathbb{R}$. Define a relation R on S by*

$$(f, g) \in R \Leftrightarrow \exists c \in \mathbb{R}, c \neq 0, \text{ such that } f(x) = cg(x) \text{ for all } x \in \mathbb{R}.$$

Prove that R is an equivalence relation.

Let $f \in S$.

Since $f(x) = (1)f(x)$ for all $x \in \mathbb{R}$, $(f, f) \in R$.

Hence, R is reflexive.

Suppose $(f, g) \in R$.

Then there exists a $c \in \mathbb{R}, c \neq 0$ such that $f(x) = cg(x)$ for all $x \in \mathbb{R}$. Since $c \neq 0$,

$$g(x) = \frac{1}{c} f(x)$$

for all $x \in \mathbb{R}$ and $\frac{1}{c} \neq 0, \frac{1}{c} \in \mathbb{R}$.

Thus $(g, f) \in R$.

Hence, R is symmetric.

Suppose $(f, g) \in R$ and $(g, h) \in R$.

Then there exist non-zero $c, d \in \mathbb{R}$ such that $f(x) = cg(x)$ and $g(x) = dh(x)$ for all $x \in \mathbb{R}$.

Hence, $f(x) = cdh(x)$ for all $x \in \mathbb{R}$.

Since c and d are non-zero, cd is non-zero, and $cd \in \mathbb{R}$, so $(f, h) \in R$.

Thus, R is transitive, and so R is an equivalence relation. ■

16. *Let A and B be sets.*

Let f and g be functions from A to B.

Prove that if $f \cap g \neq \varnothing$, then $f \setminus g$ is not a function from A to B.

Proof: Suppose $f \cap g \neq \varnothing$.

Suppose $(a, b) \in f \cap g$ (note that this is the unique pair in f with first element a).

Then $(a, b) \notin f \setminus g$.

Since f is a function, there is no element $(x, y) \in f \setminus g$ such that $x = a$.

Hence $f \setminus g$ is not a function. ■

17. *Let $n \in \mathbb{Z}_{>0}$.*

Use induction to prove $\displaystyle\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$.

Proof: Let $P(n)$ be the statement "$\displaystyle\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$".

Since

$$\sum_{i=1}^{1} \frac{1}{(2i-1)(2i+1)} = \frac{1}{3} = \frac{1}{2(1)+1},$$

P(1) is true.

Suppose there exists a $k > 0$ such that $P(k)$ is true.

Then

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{(2i-1)(2i+1)} &= \sum_{i=1}^{k} \frac{1}{(2i-1)(2i+1)} + \frac{1}{(2(k+1)-1)(2(k+1)+1)} \\
&= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} \text{ (by the induction hypothesis)} \\
&= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} \\
&= \frac{k(2k+3)+1}{(2k+1)(2k+3)} \\
&= \frac{2k^2+3k+1}{(2k+1)(2k+3)} \\
&= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} \\
&= \frac{k+1}{2k+3}
\end{aligned}$$

Hence $P(k+1)$ is true, so $P(k)$ implies $P(k+1)$.

Hence, by induction, $P(k)$ is true for all $k > 0$. ■