MATH 300 C, Winter 2015
Midterm I Study Problems Solutions

1. Prove that, for all $x \in \mathbb{Z}$, if $x^2 - 1$ is divisible by 8, then x is odd.

   Proof: Suppose x is an even integer.

   Then $x = 2k$ for some integer k.

   Then $x^2 - 1 = (2k)^2 - 1 = 4k^2 - 1$.

   Suppose $x^2 - 1$ is divisible by 8.

   Then there exists an integer m such that $4k^2 - 1 = 8m$.

   Then $1 = 8m - 4k^2 = 4(2m - k^2)$.

   Since $2m - k^2 \in \mathbb{Z}$, 4 divides 1.

   However, 4 does not divide 1 since $0 < 1 < 4$, so we have a contradiction.

   Hence, the assumption that $x^2 - 1$ is divisible by 8 is false.

   Thus $x^2 - 1$ is not divisible by 8.

   Hence, if $x^2 - 1$ is divisible by 8, then x is odd. ■

2. Let a and b be integers. Prove that $x = a^2 + ab + b$ is odd iff a is odd or b is odd.

   Proof: Suppose a and b are integers.

   We can consider four cases.

   (a) Suppose a is even and b is even.
   Then $a = 2k$ and $b = 2m$ for some integers k and m.
   Then $x = 4k^2 + 4km + 2m = 2(2k^2 + 2km + m)$.
   Since $2k^2 + 2km + m$ is an integer (by Closure), x is even.

   (b) Suppose a is even and b is odd.
   Then $a = 2k$ and $b = 2m + 1$ for some integers k and m.
   Then $x = 4k^2 + 2k(2m + 1) + 2m + 1 = 2(2k^2 + k(2m + 1) + m) + 1$.
   Since $2k^2 + k(2m + 1) + m$ is an integer (by Closure), x is odd.

   (c) Suppose a is odd and b is even.
   Then $a = 2k + 1$ and $b = 2m$ for some integers k and m.
   Then

$$\begin{aligned} x &= (2k + 1)^2 + (2k + 1)(2m) + 2m \\ &= 4k^2 + 4k + 1 + (2k + 1)(2m) + 2m \\ &= 2(2k^2 + 2k + m(2k + 1) + m) + 1. \end{aligned}$$

   Since $2k^2 + 2k + m(2k + 1) + m$ is an integer (by Closure), x is odd.

   (d) Suppose a is odd and b is odd.
   Then $a = 2k + 1$ and $b = 2m+!$ for some integers k and m.
   Then

$$\begin{aligned} x &= (2k + 1)^2 + (2m + 1)(2k + 1) + b \\ &= 4k^2 + 4k + 1 + 4mk + 2m + 2k + 1 + 2m + 1 \\ &= 2(2k^2 + 2mk + 3k + 2m + 1) + 1. \end{aligned}$$

   Since $(2k^2 + 2mk + 3k + 2m + 1)$ is an integer (by Closure), x is odd.

The first case shows that if a and b are even, then x is even; hence, if x is odd, then a is odd or b is odd.

The other three cases show that if a or b is odd, then x is odd.

Hence, x is odd if and only if a is odd or b is odd. ■

3. Let a and b be integers. Prove that $a(b+a+1)$ is odd iff a and b are both odd.

   Proof: Let a and b be integers.

   Suppose a is even.

   Then $a = 2k$ for some $k \in \mathbb{Z}$.

   Then $a(b+a+1) = 2k(b+a+1)$, and since a, b, and k are integers, $k(b+a+1)$ is an integer (by Closure).

   Hence, $2 \mid a(b+a+1)$, i.e., $a(b+a+1)$ is even.

   Suppose a is odd and b is even.

   Then $a = 2k+1$ and $b = 2m$ for some integers k and m.

   Then $a(b+a+1) = a(2k+2m+1+1) = a(2k+2m+2) = 2a(k+m+1)$.

   Since a, k, and m are integers, $a(k+m+1)$ is an integer (by Closure), and so $2 \mid a(b+a+1)$, i.e., $a(b+a+1)$ is even.

   Suppose a is odd and b is odd.

   Then $a = 2k+1$ and $b = 2m+1$ for some integers k and m.

   Then

$$\begin{aligned} a(b+a+1) &= (2k+1)(2k+2m+3) \\ &= 4k^2 + 4mk + 8k + 2m + 3 \\ &= 2(2k^2 + 4mk + 4k + m + 1) + 1. \end{aligned}$$

   Since k and m are integers, $2k^2 + 4mk + 4k + m + 1$ is an integer (by Closure), and so $a(b+a+1)$ is odd.

   Thus, $a(b+a+1)$ is odd iff a and b are both odd.■

4. Prove or give a counterexample for each of the following statements.

   (a) For all integers a and b, if $a|b$ and $b|a$, then $a = b$ or $a = -b$.
   Proof: Let a and b be non-zero integers.
   Suppose $a|b$ and $b|a$.
   Then there exist integers k and m such that $b = ak$ and $a = bm$.
   Then, by the Substitution of Equals axiom, we have $b = bmk$. Hence, $b - bmk = 0$, and so $b(1 - mk) = 0$.
   As $b \neq 0$, we may conclude that $1 - mk = 0$ (by EPI 6).
   That is, $mk = 1$ (by Substitution of Equals).
   Hence, $k = 1$ or $k = -1$ (by EPI 16), so $b = a$ or $b = -a$. ■

(b) For all integers m and n, if $n+m$ is odd, then $n \neq m$.
Proof: Let m and n be integers.
Suppose $n = m$. Then $n+m = 2m$ which is even.
Hence, $n = m$ implies $n+m$ is even, and so if $n+m$ is odd, then $n \neq m$. ■

5. Let A, B, and C be sets. Prove that $A \cap B = A \setminus (A \setminus B)$.

   Proof: Let A, B, and C be sets.

   Suppose $x \in A \cap B$.

   Then $x \in A$ and $x \in B$.

   Hence, $x \notin A \setminus B$.

   Since $x \in A$, we conclude $x \in A \setminus (A \setminus B)$.

   Thus, $x \in A \cap B$ implies $x \in A \setminus (A \setminus B)$, so $A \cap B \subseteq A \setminus (A \setminus B)$.

   Now, suppose $x \in A \setminus (A \setminus B)$.

   Then $x \in A$.

   Suppose $x \notin B$.

   Then $x \in A \setminus B$, and hence $x \notin A \setminus (A \setminus B)$.

   This is a contradiction, since $x \in A \setminus (A \setminus B)$.

   Thus $x \in B$, and so $x \in A \cap B$.

   Hence, $x \in A \setminus (A \setminus B)$ implies $x \in A \cap B$.

   Therefore $A \setminus (A \setminus B) \subseteq A \cap B$.

   Thus, $A \cap B \subseteq A \setminus (A \setminus B)$ and $A \setminus (A \setminus B) \subseteq A \cap B$, and so $A \cap B = A \setminus (A \setminus B)$. ■

6. Let A, B and C be sets. Prove that $(A \cup B) \setminus (A \cup C) = B \setminus (A \cup C)$.

   Proof: Let A, B and C be sets.

   Suppose $x \in (A \cup B) \setminus (A \cup C)$.

   Then $x \in A \cup B$ and $x \notin A \cup C$.

   Hence, $x \notin A$.

   Since $x \in A \cup B$, we conclude that $x \in B$.

   Hence, $x \in B \setminus (A \cup C)$.

   Thus, $x \in (A \cup B) \setminus (A \cup C)$ implies $x \in B \setminus (A \cup C)$, and so $(A \cup B) \setminus (A \cup C) \subseteq B \setminus (A \cup C)$.

   Now, suppose $x \in B \setminus (A \cup C)$.

   Then $x \in B$, so $x \in A \cup B$.

   Also, $x \notin A \cup C$, and so $x \in (A \cup B) \setminus (A \cup C)$.

   Hence, $x \in B \setminus (A \cup C)$ implies $x \in (A \cup B) \setminus (A \cup C)$, so $B \setminus (A \cup C) \subseteq x \in (A \cup B) \setminus (A \cup C)$.

   Thus, $(A \cup B) \setminus (A \cup C) \subseteq B \setminus (A \cup C)$ and $B \setminus (A \cup C) \subseteq x \in (A \cup B) \setminus (A \cup C)$.

   Hence, $(A \cup B) \setminus (A \cup C) = B \setminus (A \cup C)$. ■

7. Let A, B and C be sets. Prove that $(A \setminus B) \setminus C = A \setminus (B \cup C)$.

Proof: Let A, B and C be sets.

Suppose $x \in (A \setminus B) \setminus C$.

Then $x \in A$, $x \notin B$ and $x \notin C$.

Suppose $x \in B \cup C$.

Then $x \in B$ or $x \in C$.

This is a contradiction since $x \notin B$ and $x \notin C$.

Thus, $x \notin B \cup C$.

Hence, $x \in A \setminus (B \cup C)$.

So, $x \in (A \setminus B) \setminus C$ implies $x \in A \setminus (B \cup C)$, and hence $(A \setminus B) \setminus C \subseteq x \in A \setminus (B \cup C)$.

Now, suppose $x \in A \setminus (B \cup C)$.

Then $x \in A$ and $x \notin B \cup C$.

Then $x \notin B$ and $x \notin C$, and so $x \in A \setminus B$, and $x \in (A \setminus B) \setminus C$.

Thus, $x \in A \setminus (B \cup C)$ implies $x \in (A \setminus B) \setminus C$, and so $A \setminus (B \cup C) \subseteq (A \setminus B) \setminus C$.

Hence, $(A \setminus B) \setminus C \subseteq x \in A \setminus (B \cup C)$ and $A \setminus (B \cup C) \subseteq (A \setminus B) \setminus C$.

Therefore, $(A \setminus B) \setminus C = A \setminus (B \cup C)$. ■

8. Let A, B, and C be sets. Prove that $A \cup C \subseteq B \cup C$ iff $A \setminus C \subseteq B \setminus C$.

Proof: Let A, B, and C be sets.

Suppose $A \cup C \subseteq B \cup C$.

Suppose $x \in A \setminus C$.

Then $x \in A$ and $x \notin C$.

Then $x \in A \cup C$, so $x \in B \cup C$.

Since $x \notin C$, we conclude that $x \in B$.

Hence, $x \in B$ and $x \notin C$, i.e., $x \in B \setminus C$.

Thus, $x \in A \setminus C$ implies $x \in B \setminus C$, so $A \setminus C \subseteq B \setminus C$.

Therefore, $A \cup C \subseteq B \cup C$ implies $A \setminus C \subseteq B \setminus C$.

Now, suppose $A \setminus C \subseteq B \setminus C$.

Suppose $x \in A \cup C$.

Then $x \in A$ or $x \in C$.

Suppose $x \in C$.

Then $x \in B \cup C$.

Suppose $x \notin C$.

Then $x \in A$, and so $x \in A \setminus C$, and hence $x \in B \setminus C$.

So $x \in B$, and hence $x \in B \cup C$.

Hence, $x \in B \cup C$, and so $x \in A \cup C$ implies $x \in B \cup C$.

That is, $A \cup C \subseteq B \cup C$.

Thus, $A \setminus C \subseteq B \setminus C$ implies $A \cup C \subseteq B \cup C$.

Thus, $A \cup C \subseteq B \cup C$ iff $A \setminus C \subseteq B \setminus C$. ■

9. Write out the set (i.e., express the set by listing its elements) given by the expression

$$\mathcal{P}(\{1,2,3\}) \cap \mathcal{P}(\{2,3,4\}).$$

$$\mathcal{P}(\{1,2,3\}) \cap \mathcal{P}(\{2,3,4\}) = \{\varnothing, \{2\}, \{3\}, \{2,3\}\}.$$

10. Let A and B be sets. Prove that

$$\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B).$$

Proof: Let A and B be sets.

Suppose $x \in \mathcal{P}(A) \cup \mathcal{P}(B)$.

Then $x \in \mathcal{P}(A)$ or $x \in PP(B)$.

Suppose $x \in \mathcal{P}(A)$.

Then $x \subseteq A$.

Suppose $y \in x$.

Then $y \in A$, and so $y \in A \cup B$.

Thus, $y \in x$ implies $y \in A \cup B$, so $x \subseteq A \cup B$.

Hence, $x \in \mathcal{P}(A \cup B)$.

Suppose $x \in \mathcal{P}(B)$.

Then $x \subseteq B$.

Suppose $y \in x$.

Then $y \in B$, and so $y \in A \cup B$.

Thus, $y \in x$ implies $y \in A \cup B$, so $x \subseteq A \cup B$.

Hence, $x \in \mathcal{P}(A \cup B)$.

Therefore, $x \in \mathcal{P}(A) \cup \mathcal{P}(B)$ implies $x \in \mathcal{P}(A \cup B)$, so $\mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. ■

11. Let A and B be sets. Prove that $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

Proof: Suppose $x \in \mathcal{P}(A \cap B)$.

Then $x \subseteq A \cap B$.

Suppose $z \in x$.

Then $z \in A \cap B$, so $z \in A$ and $z \in B$.

Thus, $z \in x$ implies $z \in A$, so $x \subseteq A$, and $z \in x$ implies $z \in B$, so $x \subseteq B$.

Hence, $x \in \mathcal{P}(A)$ and $x \in \mathcal{P}(B)$, so $x \in \mathcal{P}(A) \cap \mathcal{P}(B)$.

Thus, $x \in \mathcal{P}(A \cap B)$ implies $x \in PP(A) \cap \mathcal{P}(B)$, so $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$.

Now, suppose $x \in \mathcal{P}(A) \cap \mathcal{P}(B)$.

Then $x \in \mathcal{P}(A)$ and $x \in \mathcal{P}(B)$, i.e., $x \subseteq A$ and $x \subseteq B$.

Suppose $y \in x$.

Then $y \in A$ and $y \in B$, so $y \in A \cap B$.

Thus, $y \in x$ implies $y \in A \cap B$, so $x \subseteq A \cap B$, i.e., $x \in \mathcal{P}(A \cap B)$.

Hence, $x \in \mathcal{P}(A) \cap \mathcal{P}(B)$ implies $x \in \mathcal{P}(A \cap B)$, and so $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$.

Thus, $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$ and $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$.

Therefore, $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$. ■

12. Let A and B be sets. Prove that $A = B$ iff $\mathcal{P}(A) = \mathcal{P}(B)$.

Proof:

Let A and B be sets.

Suppose $A = B$. Then $\mathcal{P}(A) = \mathcal{P}(B)$.

Now, suppose $\mathcal{P}(A) = \mathcal{P}(B)$.

Suppose $x \in A$.

Then $\{x\} \in \mathcal{P}(A)$, so $\{x\} \in \mathcal{P}(B)$.

Hence, $\{x\} \subseteq B$.

Since $x \in \{x\}$, $x \in B$.

Hence, since $x \in A$ implies $x \in B$, $A \subseteq B$.

Now, suppose $y \in B$.

Then $\{y\} \in \mathcal{P}(B)$, so $\{y\} \in \mathcal{P}(A)$.

Hence, $\{y\} \subseteq A$.

Since $y \in \{y\}$, $y \in A$.

Hence, since $y \in B$ implies $y \in A$, $B \subseteq A$.

Thus $A = B$.

Therefore, $A = B$ iff $\mathcal{P}(A) = \mathcal{P}(B)$. ■

13. Let A and B be sets. Prove that $A \cap B = \varnothing$ if and only if $\mathcal{P}(A) \cap \mathcal{P}(B) = \{\varnothing\}$. (Bonus: think about proving this with and without using # 11.)

Proof (without using # 11): Let A and B be sets.

Suppose $A \cap B \neq \varnothing$.

Then there exists x such that $x \in A$ and $x \in B$.

Let $S = \{x\}$, the set containing x as its only element.

Since $x \in A$, $S \subseteq A$, so $S \in \mathcal{P}(A)$.

Since $x \in B$, $S \subseteq B$, so $S \in \mathcal{P}(B)$.

Thus, $S \in \mathcal{P}(A) \cap \mathcal{P}(B)$, and $S \neq \varnothing$, so $\mathcal{P}(A) \cap \mathcal{P}(B) \neq \{\varnothing\}$.

Suppose $\mathcal{P}(A) \cap \mathcal{P}(B) \neq \{\varnothing\}$.

Since $\varnothing \in \mathcal{P}(A) \cap \mathcal{P}(B)$, we can conclude that there is a set $S \in \mathcal{P}(A) \cap \mathcal{P}(B)$ and $S \neq \varnothing$.

Hence, there is an element $x \in S$.

Since $S \in \mathcal{P}(A)$, $S \subseteq A$, and so $x \in A$.

Since $S \in \mathcal{P}(B)$, $S \subseteq B$, and so $x \in B$.

Thus, $x \in A \cup B$, so $A \cup B \neq \varnothing$.

Hence, $A \cap B = \varnothing$ if and only if $\mathcal{P}(A) \cap \mathcal{P}(B) = \{\varnothing\}$.■