Homework 5 - Math 300 C - Spring 2015 - Dr. Matthew Conroy

1. Prove that there are infinitely many positive integers that are not the sum of two cubes (hint: look at the situation modulo 7).

2. Prove that $20 \mid 3^{5427} - 7$.

3. Prove that $35 \mid 14^{7800} - 21$.

4. Suppose $f : A \to C$ and $g : B \to C$. Prove that if $A \cap B = \varnothing$, then $f \cup g : (A \cup B) \to C$.

5. Suppose R is a relation on a set A. Is it possible that R is both a function (i.e., $R : A \to A$) and an equivalence relation? Answer this question as specifically as possible by completing and proving the statement "R is a function and an equivalence relation iff ...".

6. Let S and T be sets and $f : S \to T$. Define a relation R on S by

$$(a, b) \in R \Leftrightarrow f(a) = f(b).$$

   Prove that R is an equivalence relation.

7. Let A, B and C be sets. Let $f : A \to B$ and $g : B \to C$.

   (a) Prove that if f and g are onto, then $g \circ f$ is onto.
   (b) Prove that if $g \circ f$ is onto, then g is onto.
   (c) If $g \circ f$ is onto, is f necessarily onto? Prove your answer.

8. Let A be the set of subsets of $\mathbb{R}$. Define a function $f : \mathbb{R} \to A$ by

$$f(x) = \{z \in \mathbb{R} : |z| > x\}.$$

   Is f one-to-one? Is f onto?

9. Suppose A, B and C are sets. Suppose $f : A \to B$ and $g : B \to C$.

   (a) Prove that if f is onto and g is not one-to-one, then $g \circ f$ is not one-to-one.
   (b) Prove that if f is not onto and g is one-to-one, then $g \circ f$ is not onto.