Homework 4 - Math 300 C Spring 2015 - Dr. Matthew Conroy

1. Prove the following theorem: Suppose $\mathcal{R}$ and $\mathcal{S}$ are families of sets. If $\mathcal{R} \subseteq \mathcal{S}$, then $\cup\mathcal{R} \subseteq \cup\mathcal{S}$.

2. Prove the following theorem: Suppose $\mathcal{R}$ and $\mathcal{S}$ are families of sets, and $\mathcal{R} \neq \varnothing$ and $\mathcal{S} \neq \varnothing$. If $\mathcal{R} \subseteq \mathcal{S}$, then $\cap\mathcal{S} \subseteq \cap\mathcal{R}$.

3. Let $A = \{1, 2, 3\}$. List all equivalence relations on the set A. (Hint: there are five of them).

4. For each of the following relations, answer the following questions. Is it reflexive? Is it symmetric? Is it transitive? Support all answer with proof. Conclude whether or not the relation is an equivalence relation.

   (a) Define a relation $R \subseteq \mathbb{R} \times \mathbb{R}$ by
   $$(x, y) \in R \text{ iff } x < y.$$

   (b) Define a relation $R \subseteq \mathbb{R} \times \mathbb{R}$ by
   $$(x, y) \in R \text{ iff } x \leq y.$$

   (c) Define a relation $R \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ by
   $$((x_1, y_1), (x_2, y_2)) \in R \text{ iff the distance from } (x_1, y_1) \text{ to } (x_2, y_2) \text{ is an integer}.$$

   (d) Let A be the set of all functions from $\mathbb{R}$ to $\mathbb{R}$. Define a relation $R \subseteq A \times A$ by
   $$(f, g) \in R \text{ iff there exists a } k \in \mathbb{R} \text{ such that } f(x) = g(x) + k \text{ for all } x.$$

5. Prove the following theorem: Suppose A is a set and that $R \subseteq A \times A$ and $S \subseteq A \times A$ are equivalence relations. Then $R \cap S$ is an equivalence relation.

6. Prove that the union of two equivalence relations on a set A need not be an equivalence relation. (Hint: you can use the relations you listed in problem 3).

7. Let $a, b \in \mathbb{Z}$. Let $m \in \mathbb{Z}_{>0}$.

   We say a is **congruent** to b mod m iff $m|(a - b)$.

   If a is congruent to b mod m, we write $a \equiv b \pmod{m}$.

   Prove that if $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $(a + c) \equiv (b + d) \pmod{m}$ and $ac \equiv bd \pmod{m}$.