Homework 4 - Math 301 A - Spring 2014 - Dr. Matthew Conroy
You should read Harold, section 3.3.

1. Prove that there are **not** arbitrarily long runs of consecutive square-free integers.

2. Let $k \in \mathbb{Z}$, $k > 0$. Prove that a positive integer n is divisible by 2^k if and only if the number represented by the right-most k digits is divisible by 2^k.

3. Prove that an integer n is divisible by 11 if and only if the sum of its even-place digits is congruent to the sum of its odd-place digits modulo 11. (For example, 75849 is not divisible by 11 since $9+8+7 \equiv 2 \pmod{11}$ while $4+5 \equiv 9 \pmod{11}$).

4. (You might want to do problem 6 before you do this one.) A man has a large number of doughnuts. He sorts them evenly into 7 piles. He then eats one doughnut and sorts the rest evenly into 5 piles. He then eats two more doughnuts and sorts the rest evenly into 11 piles. He then eats one more doughnut and sorts the rest evenly 6 piles.

   (a) What is the smallest number of doughnuts the man could have started with?

   (b) If you also know that he started with at least one million doughnuts, what is the smallest number he could have started with?

5. Show that every integer x satisfies at least one of the following congruences:

   $x \equiv 0 \pmod{2}, x \equiv 0 \pmod{3}, x \equiv 1 \pmod{4}, x \equiv 1 \pmod{6}, x \equiv 3 \pmod{8}, x \equiv 11 \pmod{12}$

6. Solve each of the following sets of congruences.

   (a)
$$x \equiv 1 \pmod{15}$$
$$x \equiv 2 \pmod{14}$$
$$x \equiv 3 \pmod{11}$$

   (b)
$$x \equiv 5 \pmod{7}$$
$$x \equiv 3 \pmod{10}$$
$$x \equiv 1 \pmod{19}$$
$$x \equiv 2 \pmod{33}$$

   (c)
$$x \equiv 1 \pmod{10}$$
$$x \equiv 1 \pmod{15}$$
$$x \equiv 6 \pmod{35}$$

   (Note: the moduli are not relatively prime in this last one, so be careful how you proceed.)