Below are some extra examples for Math 300. These examples are to augment the lectures and help you do some of the assigned homework problems.

1. To show \((P \land R) \lor (\neg R \land (P \lor Q))\) is equivalent to \(P \lor (\neg R \land Q)\):

\[
(P \land R) \lor (\neg R \land (P \lor Q))
\]

is equivalent to

\[
(P \land R) \lor (\neg R \land P) \lor (\neg R \land Q)
\]

(distributive and associative laws)

is equivalent to

\[
(((P \land R) \lor \neg R) \land ((P \land R) \lor P)) \lor (\neg R \land Q)
\]

(distributive law)

is equivalent to

\[
(((P \land R) \lor \neg R) \land P) \lor (\neg R \land Q)
\]

(absorbtion)

is equivalent to

\[
((\neg R \lor P) \land (\neg R \lor R) \land P) \lor (\neg R \land Q)
\]

(distributive law)

is equivalent to

\[
P \lor (\neg R \land Q)
\]

(absorbtion and tautology)

2. Enumeration of all logical connective possibilities.

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3. After working with the binary connectives, \(\lor\) and \(\land\), we might wonder if there are ternary connectives. First, some notation. We could define new notation for the binary connectives. For instance, we could define \((A, B)_1\) to be equivalent to \((A \land B)\) and \((A, B)_2\) to be equivalent to \((A \lor B)\).

From the table above, we could define 16 different such connectives, i.e., \((A, B)_1, (A, B)_2, ..., (A, B)_{16}\). Extending this notation, we could define a ternary connective \((A, B, C)_1\) with the following truth table:
Now, here’s the interesting part: we can show this is equivalent to an expression using only $A$, $B$, $C$, $\lor$, $\land$ and $\neg$. Here’s how.

Consider just the rows that have $T$ in the right-most column. For each such row, consider the expression

$$(\neg A \land (\neg B \land (\neg C))$$

where the $\neg$ if there is there is a $F$ in that statements column, and no $\neg$ otherwise.

For instance, for the first row, we have

$$\neg A \land \neg B \land \neg C$$

and for the fifth row, we have

$$A \land \neg B \land \neg C$$

We can create these expressions for each needed row. If we string these expressions together with $\lor$, we will have an expression that is equivalent to $(A, B, C)_1$:

$$(\neg A \land \neg B \land \neg C) \lor (A \land \neg B \land \neg C) \lor (A \land \neg B \land \neg C) \lor (A \land \neg B \land \neg C) \lor (A \land \neg B \land \neg C)$$

(There is certainly a chance that this could be simplified, of course.)

This process shows that any ternary connective can be expressed using only $\lor$, $\land$ and $\neg$. In fact, we can extend this to any number of input statements: more columns on the left in the table would not have effected our procedure at all. So even though we can imagine complex many-input logical constructs, they are all equivalent to expressible with just $\lor$, $\land$ and $\neg$.

(And as we’ll see elsewhere, you can get by with just $\lor$ and $\neg$, or just $\land$ and $\neg$.)

4. Suppose we wish to show that $\exists x(P(x) \rightarrow Q(x))$ is equivalent to $\forall xP(x) \rightarrow \exists xQ(x)$.

By the conditional laws (given on page 47 of Velleman),

$$\exists x(P(x) \rightarrow Q(x))$$

is equivalent to

$$\exists x(\neg P(x) \lor Q(x)).$$

(1)

Now, any statement $\exists x(R(x) \lor S(x))$ means that there exists an $x$ such that $R(x)$ or $S(x)$ is true. That is equivalent to saying that there exists an $x$ such that $R(x)$ is true, or there exists an $x$ such that $S(x)$ is true. That is,

$$\exists x(R(x) \lor S(x))$$

is equivalent to $\exists xR(x) \lor \exists xS(x)$

Thus, expression (1) is equivalent to

$$\exists x\neg P(x) \lor \exists xQ(x)$$
and this is equivalent, by our quantifier negation laws, to
\[ \neg \forall x P(x) \lor \exists x Q(x) \]
and this is equivalent to
\[ \forall x P(x) \rightarrow \exists x Q(x) \]
by our conditional laws.
Along these same lines, note that
\[ \forall x (R(x) \land S(x)) \]
says that for all x, R(x) and S(x) are true. This is equivalent to saying that for all x R(x) is true and for all x S(x) is true (the latter just takes longer to say!). Hence, \( \forall x (R(x) \land S(x)) \) is equivalent to
\[ \forall x R(x) \land \forall x S(x) . \]

5. Why can we prove by cases?
Suppose we want to show \( A \rightarrow S \).
Suppose we know \( B \) or \( C \) is true (e.g., for an integer \( n \), we might set \( B = "n \) is even" and \( C = "n \) is odd").
Then we use:

\[
((A \land B) \rightarrow S) \land ((A \land C) \rightarrow S) \\
\leftrightarrow (\neg (A \land B) \lor S) \land (\neg (A \land C) \lor S) \\
\leftrightarrow (\neg A \lor \neg B \lor S) \land (\neg A \lor \neg C \lor S) \\
\leftrightarrow ((\neg A \lor \neg B) \land (\neg A \lor \neg C)) \lor S \\
\leftrightarrow (\neg A \lor (\neg B \land \neg C)) \lor S \\
\leftrightarrow \neg A \lor S \\
\leftrightarrow A \rightarrow S .
\]

Note we use the fact that \( \neg B \land \neg C \) is false here.

6. An example of a uniqueness argument.
Theorem: Let \( S \) be a set. For every \( A \in \mathcal{P}(S) \), there is a unique \( B \in \mathcal{P}(S) \) such that for every \( C \in \mathcal{P}(S) \), \( C \setminus A = C \cap B \).

Proof. Let \( S \) be a set. Let \( A \in \mathcal{P}(S) \). Let \( B = S \setminus A \). Then, let \( C \in \mathcal{P}(S) \).
Suppose \( x \in C \setminus A \). So, \( x \in C \) and \( x \notin A \), so \( x \in B \).
Hence, \( x \in B \cap C \). Thus, \( C \setminus A \subseteq C \cap B \).
Suppose \( x \in C \cap B \). Then \( x \in C \) and \( x \in B \), so \( x \notin A \).
Hence, \( x \in C \setminus A \). Thus, \( C \cap B \subseteq C \setminus A \).
Thus, \( C \cap B = C \setminus A \).
Thus there exists a set with the required property, namely \( B = S \setminus A \).
To show uniqueness, suppose a set \( D \) in \( \mathcal{P}(S) \) also has the required property.
Then for all \( C \in \mathcal{P}(S) \), \( C \cap D = C \setminus A \).
Since for all $C \in \mathcal{P}(S)$, $C \cap B = C \setminus A$, we have that for all $C \in \mathcal{P}(S)$,
\[ C \cap B = C \cap D. \]

In particular, if we let $C = B$, we have $B \cap B = B \cap D$, i.e., $B = B \cap D$. This shows that $B \subseteq D$, since if $x \in B$, $x$ is also in $D$.

On the other hand, if we let $C = D$, we have $D \cap D = D \cap B$, i.e., $D = D \cap B$. This shows that $D \subseteq B$, since if $x \in D$, $x$ is also in $B$.

Hence, $D = B$. And so the choice of $B$ is unique.

7. A surprising bijection.

Let’s consider the intervals of real numbers $A = (0, 1]$ and $B = (0, 1)$. Since both $A$ and $B$ contain an infinite number of elements, and $B$ is simply $A$ with one element (1) removed, it would be surprising if $A$ and $B$ were not equinumerous. However, if we try to find a bijection from one set to the other using combinations of our standard functions (e.g. trig functions, rational functions, etc.), we find that a bijection eludes us.

It turns out that it is possible, but it requires a surprising bit of cleverness. First, consider this function from $A$ to $B$:
\[ f(x) = \begin{cases} 
1/2 & \text{if } x = 1, \\
x & \text{if } x \neq 1.
\end{cases} \]

Notice that this almost works. It maps the set $(0, 1]$ onto the set $(0, 1)$, but it is not one-to-one: $f(1) = f(1/2)$.

To fix this, we might try sending $1/2$ to something else:
\[ g(x) = \begin{cases} 
1/3 & \text{if } x = 1/2, \\
1/2 & \text{if } x = 1, \\
x & \text{if } x \neq 1 \text{ and } x \neq 1/2.
\end{cases} \]

Again, $g$ maps the set $(0, 1]$ onto $(0, 1)$ and now $f(1) \neq f(1/2)$, but now $f(1/2) = f(1/3)$ so it is again not one-to-one.

One more try: let’s send $1/3$ to something else:
\[ h(x) = \begin{cases} 
1/4 & \text{if } x = 1/3, \\
1/3 & \text{if } x = 1/2, \\
1/2 & \text{if } x = 1, \\
x & \text{if } x \neq 1, x \neq 1/2, \text{ and } x \neq 1/3.
\end{cases} \]

This function maps the set $(0, 1]$ onto $(0, 1)$ and $h(1) \neq h(1/2) \text{ and } h(1/2) \neq h(1/3)$, but now $h(1/3) = h(1/4)$, so $h$ is not one-to-one.

Although this doesn’t seem to be working, if we extend this strategy forever, we get a function that does work.

Let $k : A \to B$ be defined like this:
\[ k(x) = \begin{cases} 
\frac{1}{n+1} & \text{if } x = 1/n \text{ for some } n \in \mathbb{Z}_{>0}, \\
x & \text{if } x \neq 1/n \text{ for all } n \in \mathbb{Z}_{>0}.
\end{cases} \]

So we have $k(1) = 1/2$, $k(1/2) = 1/3$, $k(1/3) = 1/4$, $k(1/4) = 1/5$, etc. We are basically pushing our problem down the sequence of $1/n$, and since this sequence is infinite, it becomes a non-problem.

It is not too hard to show that our $k$ function is a bijection from $A$ to $B$, and this shows that $A$ and $B$ are equinumerous.