# Axioms

Suppose $x$, $y$, and $z$ are real numbers. We will take as fact each of the following.

1. $x + y$ and $xy$ are real numbers. ($\mathbb{R}$ is *closed* under addition and multiplication.)
2. If $x = y$, then $x + z = y + z$ and $xz = yz$. (This is sometimes called *substitution of equals*.)
3. $x + y = y + x$ and $xy = yx$ (addition and multiplication are *commutative* in $\mathbb{R}$)
4. $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$ (addition and multiplication are *associative* in $\mathbb{R}$)
5. $x(y + z) = xy + xz$ (This is the *Distributive Law*.)
6. $x + 0 = 0 + x = x$ and $x \cdot 1 = 1 \cdot x = x$ (0 is the *additive identity*; 1 is the *multiplicative identity*.)
7. There exists a real number $-x$ such that $x + (-x) = (-x) + x = 0$. (That is, every real number has an *additive inverse* in $\mathbb{R}$.)
8. If $x \neq 0$, then there exists a real number $x^{-1}$ such that $x \cdot x^{-1} = x^{-1} \cdot x = 1$. (That is, every non-zero real number has a *multiplicative inverse* in $\mathbb{R}$.)
9. If $x > 0$ and $y > 0$, then $x + y > 0$ and $xy > 0$.
10. Either $x > 0$, $-x > 0$, or $x = 0$.
11. If $x$ and $y$ are integers, then $-x$, $x + y$, and $xy$ are integers. (The additive inverse of an integer is an integer and $\mathbb{Z}$ is closed under addition and multiplication.)

<u>NOTE:</u> It is not hard to prove that $\mathbb{Q}$, the set of rational numbers is closed under addition and multiplication and that every non-zero rational number has a multiplicative inverse in $\mathbb{Q}$.

DeMorgan's laws

$$\neg(P \wedge Q) \text{ is equivalent to } \neg P \vee \neg Q$$

$$\neg(P \vee Q) \text{ is equivalent to } \neg P \wedge \neg Q$$

Commutative Laws

$$P \wedge Q \text{ is equivalent to } Q \wedge P$$

$$P \vee Q \text{ is equivalent to } Q \vee P$$

Associative Laws

$$P \wedge (Q \wedge R) \text{ is equivalent to } (P \wedge Q) \wedge R$$

$$P \vee (Q \vee R) \text{ is equivalent to } (P \vee Q) \vee R$$

Idempotent Laws

$$P \wedge P \text{ is equivalent to } P$$

$$P \vee P \text{ is equivalent to } P$$

Distributive Laws

$$P \wedge (Q \vee R) \text{ is equivalent to } (P \wedge Q) \vee (P \wedge R)$$

$$P \vee (Q \wedge R) \text{ is equivalent to } (P \vee Q) \wedge (P \vee R)$$

Absorption Laws

$$P \vee (P \wedge Q) \text{ is equivalent to } P$$

$$P \wedge (P \vee Q) \text{ is equivalent to } P$$

Double Negation Law

$$\neg\neg P \text{ is equivalent to } P$$

Tautology Laws

$P \wedge$ (a tautology) is equivalent to P

$P \vee$ (a tautology) is a tautology

$\neg$(a tautology) is a contradiction

Contradiction Laws

$P \wedge$ (a contradiction) is a contradiction

$P \vee$ (a contradiction) is equivalent to P

$\neg$(a contradiction) is a tautology

# Elementary Properties of Real Numbers

The following properties of real numbers that allow us to do algebra follow from the axioms on the front page.

If $x$, $y$, $z$, $u$, and $v$ are real numbers, then:

1. $x \cdot 0 = 0$
2. If $x + z = y + z$, then $x = y$.
3. If $x \cdot z = y \cdot z$ and $z \neq 0$, then $x = y$.
4. $-x = (-1) \cdot x$
5. $(-x) \cdot y = -(x \cdot y)$
6. $(-x) \cdot (-y) = x \cdot y$
7. If $x \cdot y = 0$, then $x = 0$ or $y = 0$.
8. If $x \leq y$ and $y \leq x$, then $x = y$.
9. If $x \leq y$ and $y \leq z$, then $x \leq z$.
10. At least one of the following is true: $x \leq y$ or $y \leq x$.
11. If $x \leq y$, then $x + z \leq y + z$.
12. If $x \leq y$ and $0 \leq z$, then $xz \leq yz$.
13. If $x \leq y$ and $z \leq 0$, then $yz \leq xz$.
14. If $x \leq y$ and $u \leq v$, then $x + u \leq y + v$.
15. If $0 \leq x \leq y$ and $0 \leq u \leq v$, then $xu \leq yv$.
16. If $x \leq y$, then $-y \leq -x$.
17. $0 \leq x^2$
18. $0 < 1$
19. If $0 < x$, then $0 < x^{-1}$.
20. If $0 < x < y$, then $0 < y^{-1} < x^{-1}$.

And here are a couple of properties of integers.

21. Every integer is either even or odd, never both.
22. The only integers that divide 1 are $-1$ and $1$.