Injections, surjections, bijections, and inverses

1. Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 2x & \text{if } x \in \mathbb{Q} \\ -3x & \text{if } x \notin \mathbb{Q} \end{cases}$$

   Is f one-to-one? Is f onto? Is f^{-1} a function? State and prove a theorem.

2. Let a, b, c and d be real numbers. Suppose $cd \neq 0$ and $ad - bc \neq 0$.
   Define $f : \mathbb{R} \setminus \{-\frac{d}{c}\} \to \mathbb{R} \setminus \{\frac{a}{c}\}$ by

$$f(x) = \frac{ax+b}{cx+d}.$$

   (a) Show that f is one-to-one and onto.
   (b) Give a formula for $f^{-1}(x)$.

3. Let A, B and C be sets. Let $f : A \to B$ and $g : B \to C$.
   (a) Prove that if f and g are onto, then $g \circ f$ is onto.
   (b) Prove that if $g \circ f$ is onto, then g is onto.
   (c) If $g \circ f$ is onto, is f necessarily onto? Prove your answer.

4. Let A be the set of subsets of $\mathbb{R}$. Define a function $f : \mathbb{R} \to A$ by

$$f(x) = \{z : |z| > x\}.$$

   Is f one-to-one? Is f onto?

5. Let A and B be sets, and $f : A \to B$. Suppose f is one-to-one. Prove that there exists a subset $C \subseteq B$ such that $f^{-1} : C \to A$.

6. For each of the following pairs of sets, give a bijection from the first set to the second set.
   (a) $\mathbb{Z}$ and $\mathbb{Z} \setminus \{2, 5\}$
   (b) $(-2, \infty)$ and $(7, \infty)$
   (c) $(-\infty, 3)$ and $(-25, 18)$