Problems on functions
Relevant reading: Velleman, 5.1, 5.2, 5.3, 6.1.

1. Suppose $f : A \to C$ and $g : B \to C$. Prove that if $A \cap B = \varnothing$, then $f \cup g : (A \cup B) \to C$.

2. Suppose R is a relation on a set A. Is it possible that R is both a function and an equivalence relation? Complete and prove the statement "R is a function and an equivalence relation iff ...".

3. Let S and T be sets and $f : S \to T$. Define a relation R on S by
$$(a, b) \in R \Leftrightarrow f(a) = f(b).$$
Prove that R is an equivalence relation.

4. Define a function $f : \mathbb{R} \to \mathbb{R}$ by
$$f(x) = \begin{cases} 2x & \text{if } x \in \mathbb{Q} \\ -3x & \text{if } x \notin \mathbb{Q} \end{cases}$$
Is f one-to-one? Is f onto? Is f^{-1} a function? State and prove a theorem.

5. Let a, b, c and d be real numbers. Suppose $cd \neq 0$ and $ad - bc \neq 0$.
Define $f : \mathbb{R} \setminus \{-\frac{d}{c}\} \to \mathbb{R} \setminus \{\frac{a}{c}\}$ by
$$f(x) = \frac{ax + b}{cx + d}.$$
   (a) Show that f is one-to-one and onto.
   (b) Give a formula for $f^{-1}(x)$.

6. Let A, B and C be sets. Let $f : A \to B$ and $g : B \to C$.
   (a) Prove that if f and g are onto, then $g \circ f$ is onto.
   (b) Prove that if $g \circ f$ is onto, then g is onto.
   (c) If $g \circ f$ is onto, is f necessarily onto? Prove your answer.

7. Let A be the set of subsets of $\mathbb{R}$. Define a function $f : \mathbb{R} \to A$ by
$$f(x) = \{z : |z| > x\}.$$
Is f one-to-one? Is f onto?

8. Let A and B be sets, and $f : A \to B$. Suppose f is one-to-one. Prove that there exists a subset $C \subseteq B$ such that $f^{-1} : C \to A$.

9. Find the smallest $k \in \mathbb{Z}$ such that $n! > n^3$ for all $n \geq k$. Prove the result using induction.

10. Let n be a positive odd integer.
Use induction to prove that the sum of all positive odd integers less than or equal to n is $\left(\frac{n+1}{2}\right)^2$.