Problems on relations and equivalence relations
Relevant reading: Velleman, 4.1, 4.2, 4.6 (Velleman says a ton more about relations in 4.3-4.5, if you are interested, but you won't need to know any of that stuff for this course)

1. How many equivalence relations are there on a set with three elements? List them.
2. Let $A = \mathbb{R}$. Define a relation R on A by
$$(x, y) \in R \Leftrightarrow x < y$$
Determine whether or not R is an equivalence relation. Prove your answer.
3. Let $A = \mathbb{R}$. Define a relation R on A by
$$(x, y) \in R \Leftrightarrow x \leq y$$
Determine whether or not R is an equivalence relation. Prove your answer.
4. Let $A = \mathbb{R} \times \mathbb{R}$.
Define a relation R on A by
$$((x_1, y_1), (x_2, y_2)) \in R \Leftrightarrow \text{the distance from } (x_1, y_1) \text{ to } (x_2, y_2) \text{ is a rational number.}$$
Determine whether or not R is an equivalence relation. Prove your answer.
5. Let $a, b \in \mathbb{Z}$. Let $m \in \mathbb{Z}_{>0}$.
We say a is **congruent** to b mod m iff $m|(a - b)$.
If a is congruent to b mod m, we write
$$a \equiv b (\text{mod} m).$$
   (a) Show that the relation R on $\mathbb{Z}$ defined by
   $$R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a \equiv b (\text{mod} m)\}$$
   is an equivalence relation.
   (b) Prove that if $a \equiv b (\text{mod}\, m)$ and $c \equiv d (\text{mod}\, m)$, then
   $$(a + c) \equiv (b + d) (\text{mod}\, m)$$
   and
   $$ac \equiv bd (\text{mod}\, m)$$
6. Let $A = \mathbb{R}$.
Define a relation R by
$$(a, b) \in R \Leftrightarrow a - b \in \mathbb{Q}.$$
   (a) Show that R is an equivalence relation.
   (b) Give an example of one of the equivalence classes in A/R.
   (c) Prove that there are infinitely many equivalence classes in A/R.