Multiplying by the Conjugate

Sometimes it is useful to eliminate square roots from a fractional expression.

A way to do this is to utilize the fact that $(A+B)(A-B) = A^2 - B^2$ in order to eliminate square roots via squaring.

For instance, consider the expression

$$\frac{\sqrt{x}+x^2}{x-2}.$$

Suppose we want to eliminate the square root from the numerator (this is sometimes called *rationalizing the numerator*).

What we can do it multiply the entire expression by

$$\frac{\sqrt{x}-x^2}{\sqrt{x}-x^2}.$$

Since this is essentially equal to 1 (that is, it *is* 1 unless $x = 1$ or $x = 0$, in which case it is undefined), our resulting expression will be essentially equivalent but it will have a *different form*:

$$\frac{\sqrt{x}+x^2}{x-2} = \frac{\sqrt{x}+x^2}{x-2}\left(\frac{\sqrt{x}-x^2}{\sqrt{x}-x^2}\right) = \frac{x-x^4}{(x-2)(\sqrt{x}-x^2)}$$

The equality if true as long as $x \neq 1$ or 0.

Note that we have moved the root from the numerator to the denominator; that's what this technique does.

Here's another example:

$$\frac{x-\sqrt{x}}{s} = \frac{x-\sqrt{x}}{s}\left(\frac{x+\sqrt{x}}{x+\sqrt{x}}\right) = \frac{x^2-x}{s(x+\sqrt{x})}$$

which is true provided $x \neq 0$.

The place students see this often is when working with the *difference quotient* expression

$$\frac{f(x+h)-f(x)}{h}.$$

If $f(x)$ is a square root function, then multiplication by the conjugate can be used to simplify this expression (in particular, to eliminate the h from the denominator).

Here's an example of this.

Suppose $f(x) = \sqrt{2x-1}$.

Then

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{\sqrt{2(x+h)-1}-\sqrt{2x-1}}{h} \\
&= \frac{\sqrt{2(x+h)-1}-\sqrt{2x-1}}{h}\left(\frac{\sqrt{2(x+h)-1}+\sqrt{2x-1}}{\sqrt{2(x+h)-1}+\sqrt{2x-1}}\right) \\
&= \frac{2(x+h)-1-(2x-1)}{h(\sqrt{2(x+h)-1}+\sqrt{2x-1})} \\
&= \frac{2h}{h(\sqrt{2(x+h)-1}+\sqrt{2x-1})} \\
&= \frac{2}{(\sqrt{2(x+h)-1}+\sqrt{2x-1})}
\end{aligned}$$

provided $h \neq 0$.

We can, as well, move radicals from the denominator to the numerator:

$$\frac{1}{\sqrt{x}+1} = \frac{1}{\sqrt{x}+1}\left(\frac{\sqrt{x}-1}{\sqrt{x}-1}\right) = \frac{\sqrt{x}-1}{x-1}$$

provided $x \neq 1$, since $\sqrt{x}-1=0$ when $x=1$.