Multiplying by the Conjugate

Sometimes it is useful to eliminate square roots from a fractional expression.

A way to do this is to utilize the fact that \((A + B)(A - B) = A^2 - B^2\) in order to eliminate square roots via squaring.

For instance, consider the expression

\[
\frac{\sqrt{x} + x^2}{x - 2}.
\]

Suppose we want to eliminate the square root from the numerator (this is sometimes called \textit{rationalizing the numerator}).

What we can do it multiply the entire expression by

\[
\frac{\sqrt{x} - x^2}{\sqrt{x} - x^2}.
\]

Since this is essentially equal to 1 (that is, it is 1 unless \(x = 1\) or \(x = 0\), in which case it is undefined), our resulting expression will be essentially equivalent but it will have a \textit{different form}:

\[
\frac{\sqrt{x} + x^2}{x - 2} = \frac{\sqrt{x} + x^2}{x - 2} \left( \frac{\sqrt{x} - x^2}{\sqrt{x} - x^2} \right) = \frac{x - x^4}{(x - 2)(\sqrt{x} - x^2)}.
\]

The equality is true as long as \(x \neq 1\) or \(0\).

Note that we have moved the root from the numerator to the denominator; that’s what this technique does.

Here’s another example:

\[
\frac{x - \sqrt{x}}{s} = \frac{x - \sqrt{x}}{s} \left( \frac{x + \sqrt{x}}{x + \sqrt{x}} \right) = \frac{x^2 - x}{s(x + \sqrt{x})},
\]

which is true provided \(x \neq 0\).

The place students see this often is when working with the \textit{difference quotient} expression

\[
\frac{f(x + h) - f(x)}{h}.
\]

If \(f(x)\) is a square root function, then multiplication by the conjugate can be used to simplify this expression (in particular, to eliminate the \(h\) from the denominator).

Here’s an example of this.
Suppose \( f(x) = \sqrt{2x - 1} \).

Then
\[
\frac{f(x + h) - f(x)}{h} = \frac{\sqrt{2(x + h) - 1} - \sqrt{2x - 1}}{h} = \frac{2(x + h) - 1 - (2x - 1)}{h(\sqrt{2(x + h)} - 1 + \sqrt{2x - 1})} = \frac{2h}{h(\sqrt{2(x + h)} - 1 + \sqrt{2x - 1})} = \frac{2}{(\sqrt{2(x + h)} - 1 + \sqrt{2x - 1})}
\]
provided \( h \neq 0 \).

We can, as well, move radicals from the denominator to the numerator:
\[
\frac{1}{\sqrt{x + 1}} = \frac{1}{\sqrt{x + 1}} \left( \frac{\sqrt{x} - 1}{\sqrt{x} - 1} \right) = \frac{\sqrt{x} - 1}{x - 1}
\]
provided \( x \neq 1 \), since \( \sqrt{x} - 1 = 0 \) when \( x = 1 \).