MATH 300 A, B Spring 2012
Midterm Study Problems

1. Give a useful denial of each statement. You may use symbols like $\forall$ and $\exists$, et cetera.
   (a) For every real number x, there exists a real number y such that $x < y$.
   (b) There exists a function $f : \mathbb{R} \to \mathbb{R}$ such that, for every real number x, $f(2x) = 4f(x)$.
   (c) If x is a positive real number, then $\ln x \geq 0$ or $\frac{1}{x} > 1$.
   (d) For all real x, $x \neq 0 \Rightarrow x^2 > 0$.

2. Let P be the statement: "Every prime number is odd." Which of the following are logically equivalent to P? (Check all that apply.)

_______ n is odd implies n is prime.
_______ n is odd if n is prime.
_______ If n is odd, then n is prime.
_______ If n is prime, then n is odd.
_______ There exist numbers that are both prime and odd.
_______ No even number is prime.
_______ n is odd if and only if n is prime.

3. Let A, B, and C be sets.
   (a) Prove that $(A \cap B) \setminus C \subseteq (B \cup C) \setminus (B \cap C)$.
   (b) Give a counterexample that demonstrates that $(B \cup C) \setminus (B \cap C)$ is not necessarily a subset of $(A \cap B) \setminus C$.
   (c) Prove that if $A \subseteq B \setminus C$ then A and C are disjoint.

4. (a) Prove that, for all $x \in \mathbb{Z}$, if $x^2 - 1$ is divisible by 8, then x is odd.
   (b) Let P be the statement: "For all $x \in \mathbb{Z}$, if x is odd, then $x^2 - 1$ is divisible by 8."
      i. What is the negation of the statement P?
      ii. Which is true: P or its negation? Prove your claim.
   (c) Prove that, for all integers x, the remainder upon division by 12 of $x^2 + 1$ is 0, 1, 2, 5, or 10.

5. Prove or give a counterexample for each of the following statements.
   (a) For all real numbers x and y, $|x + y| = |x| + |y|$.
   (b) For all real numbers x and y, $|xy| = |x||y|$.
   (c) There is a natural number M such that, for every natural number $n > M$, $\frac{1}{n} < 0.002$.
   (d) For all integers a and b, if $a|b$ and $b|a$, then $a = b$ or $a = -b$.
   (e) For all integers m and n, if $n + m$ is odd, then $n \neq m$.

6. (a) Let x be an integer. Prove that if $\sqrt{2x}$ is an integer, then x is even.
   (b) Is the converse of the statement you proved in (a) true? Prove it or give a counterexample.
   (c) What can you conclude about $\sqrt{2x}$ if x is odd?

7. (a) Suppose B is a set and $\mathcal{F}$ is a family of sets. If $\bigcup \mathcal{F} \subseteq B$ then $\mathcal{F} \subseteq \mathcal{P}(B)$.
   (b) Suppose $\mathcal{F}$ and $\mathcal{G}$ are nonempty families of sets. Suppose every element of $\mathcal{F}$ is a subset of every element of $\mathcal{G}$. Prove that $\bigcup \mathcal{F} \subseteq \cap \mathcal{G}$.