1.5 #9

The problem asks us to find a statement equivalent to $P \Leftrightarrow Q$ using only $\neg$ and $\rightarrow$.

One way to do this is just to hunt around for an equivalent expression and then show that it is equivalent by displaying the truth tables of both statements.

Another way is like this.

The definition of the biconditional connector tells us that $A \Leftrightarrow B$ is equivalent to

$$(A \rightarrow B) \wedge (B \rightarrow A)$$

So the problem is what to do with that 'and'.

We know, and can easily show that $A \rightarrow B$ is equivalent to

$$\neg A \vee B \text{ which is equivalent to } \neg(A \wedge \neg B)$$

thanks to DeMorgan's laws.

This gives us a way to go from 'implies' to 'and'. By negating, we have

$$\neg(A \rightarrow B) \Leftrightarrow A \wedge \neg B$$

and, by replacing Q by $\neg Q$ and applying the double negative law, we have

$$\neg(A \rightarrow \neg B) \Leftrightarrow A \wedge B$$

By substituting $(P \rightarrow Q)$ for A and $(Q \rightarrow P)$ for B, we thus have

$$(P \rightarrow Q) \wedge (Q \rightarrow P) \Leftrightarrow \neg((P \rightarrow Q) \rightarrow \neg(Q \rightarrow P)).$$

2.2 #9

The question asks whether or not

$$\forall x(P(x) \vee Q(x)) \tag{1}$$

is equivalent to

$$\forall x P(x) \vee \forall x Q(x) \tag{2}$$

For two statements to be equivalent, they must have the same truth value. Hence, if we can find $P(x)$ and $Q(x)$ such that statement (1) is true while statement (2) is false, the statements are not equivalent.

Let the universe be the set of integers.

Let $P(x)$ be "x is even".

Let $Q(X)$ be "x is odd".

Then statement (1) says "For all integers x, x is even or x is odd." This is a true statement.

On the other hand, statement (2) says "Either 'for all integers x, x is even' or 'for all integers x, x is odd' is false: because 5 is odd, the statement "For all integers x, x is even" is false, and because 4 is even, the statement "For all integers x, x is odd" is false.

Since (1) and (2) have different truth values, statements (1) and (2) are *not* equivalent.

3.1 #5

Thereom Let $a, b \in \mathbb{R}$, $a < b < 0$. Then $a^2 > b^2$.

Proof. Let $a, b \in \mathbb{R}$, $a < b < 0$.
We will use some of the Elementary Properties of the Real Numbers.
Since $a < b$ and $a < 0$, by Property 13, we have

$$a^2 < ab.$$

Since $a < b$ and $b < 0$, by Property 13, we have

$$ab < b^2.$$

Since $a^2 < ab$ and $ab < b^2$, by Property 9, we have $a^2 < b^2$. □