Completing the Square

Completing the square is a technique for re-formatting certain algebraic expressions.

In particular, it is useful for taking quadratic expressions like

$$ax^2 + bx + c$$

and rewriting them as

$$ax^2 + bx + c = a(x - h)^2 + k.$$

There are several reasons for doing this. One reason is that it allows us to easily see what the vertex of the curve $y = ax^2 + bx + c$ is: it is the point (h, k).

The general method of completing the square can be shown like this:

$$\begin{aligned} y &= ax^2 + bx + c \\ &= a\left(x^2 + \tfrac{b}{a}x\right) + c \\ &= a\left(\left(x + \tfrac{b}{2a}\right)^2 - \left(\tfrac{b}{2a}\right)^2\right) + c \\ &= a\left(x + \tfrac{b}{2a}\right)^2 - \tfrac{b^2}{4a} + c \\ &= a\left(x - \left(-\tfrac{b}{2a}\right)\right)^2 - \tfrac{b^2}{4a} + c. \end{aligned}$$

This shows that the x-coordinate of the vertex is

$$h = -\frac{b}{2a}.$$

Let's do some examples.

1. Suppose $y = 2x^2 - 4x - 5$.

$y = 2(x^2 - 2x) - 5$	Factor the coefficient of x^2 out of the first two terms.
$y = 2((x - 1)^2 - 1) - 5$	Write the terms in the parentheses as a perfect square minus a constant.
$y = 2(x - 1)^2 - 2 - 5$	Distribute the coefficient that you factored out in the first step.
$y = 2(x - 1)^2 - 7$	Simplify and you're done.

2. Suppose $y = x^2 + 6x - 8$.

   Then $y = (x + 3)^2 - 3^2 - 8 = (x + 3)^2 - 17$.

   This shows that the vertex of this parabola is the point $(-3, -17)$.

3. Suppose $y = 3x^2 - 12x + 1$.

   Then

   $$\begin{aligned} y &= 3(x^2 - 4x) + 1 \\ &= 3\left((x - 2)^2 - 4\right) + 1 \\ &= 3(x - 2)^2 - 12 + 1 \\ &= 3(x - 2)^2 - 11. \end{aligned}$$

   This shows that the vertex of this parabola is the point $(2, -11)$.

4. Suppose $y = -4x^2 + 7x - \frac{1}{2}$.

   Then

   $$\begin{aligned} y &= -4\left(x^2 - \tfrac{7}{4}x\right) - \tfrac{1}{2} \\ &= -4\left(\left(x - \tfrac{7}{8}\right)^2 - \left(\tfrac{7}{8}\right)^2\right) - \tfrac{1}{2} \\ &= -4\left(x - \tfrac{7}{8}\right)^2 + (4)\left(\tfrac{49}{64}\right) - \tfrac{1}{2} \\ &= -4\left(x - \tfrac{7}{8}\right)^2 + \tfrac{41}{16}. \end{aligned}$$

5. In this example, we take the equation of a circle and convert it to standard form so we can see what the center and radius of the circle are.

   Consider the equation
   $$x^2 + y^2 - 6x + 14y - 104 = 0$$
   We rewrite it as
   $$x^2 - 6x + y^2 + 14y = 104$$
   and complete the square on the x and y terms independently:
   $$(x-3)^2 - 9 + (y+7)^2 - 49 = 104$$
   Moving the constants all to the right side results in the equation
   $$(x-3)^2 + (y+7)^2 = 162$$
   This shows that our equation is the equation of the circle with center $(3, -7)$ and radius $\sqrt{162}$.

Exercises

The following equations can be verified in two ways: by completing the square on the left, or expanding on the right. I recommend that you do both for all of them.

1. $x^2 - 10x + 32 = (x-5)^2 + 7$.
2. $3x^2 - 12x + 1 = 3(x-2)^2 - 11$.
3. $4x^2 - \frac{8}{3}x + \frac{25}{9} = 4\left(x - \frac{1}{3}\right)^2 + \frac{7}{3}$
4. $-x^2 - 2x - 4 = -(x+1)^2 - 3$.
5. $-\frac{2}{3}x^2 + \frac{20}{3}x - \frac{356}{21} = -\frac{2}{3}(x-5)^2 - \frac{2}{7}$.
6. $-5x^2 - 110x - 533 = -5(x+11)^2 + 72$.

Verify the following statements:

1. The circle $x^2 + y^2 - 6x - 8y = 375$ has center $(3, 4)$ and radius 20.
2. The circle $x^2 + y^2 + \frac{1}{2}x - y - \frac{59}{16} = 0$ has center $(-\frac{1}{4}, \frac{1}{2})$ and radius 2.
3. The circle $x^2 + 2x + y^2 + 12y + \frac{123}{4} = 0$ has center $(-1, -6)$ and radius $\frac{5}{2}$.