Worksheet

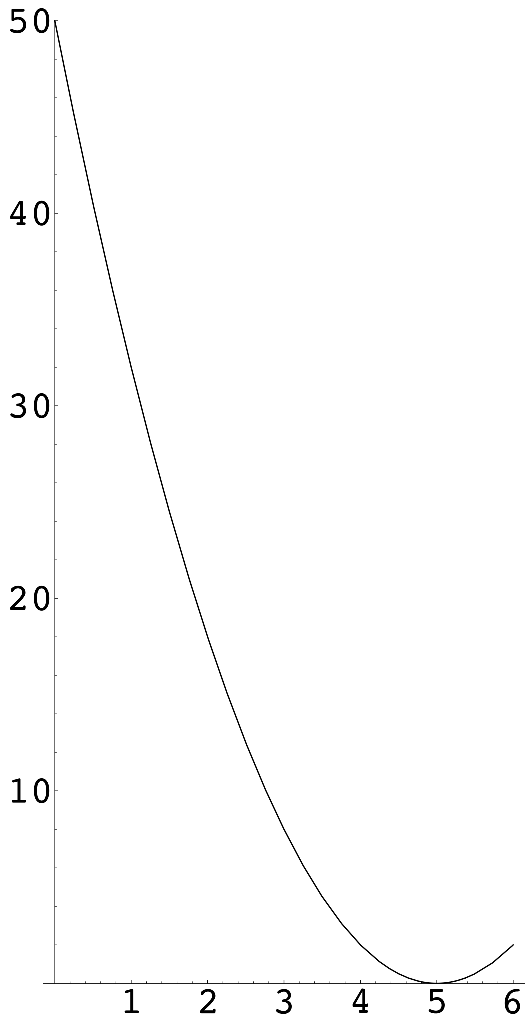
Average and Instantaneous Velocity — Math 124

Introduction

In this worksheet, we introduce what are called the *average and instantaneous velocity* in the context of a specific physical problem: A golf ball is hit toward the cup from a distance of 50 feet. Assume the distance from the ball to the cup at time t seconds is given by the function

$$d(t) = 50 - 20t + 2t^2.$$

The graph of $y = d(t)$ appears below.



1. Does the ball reach the cup? If so, when? (Answer this question two ways: by using algebra, and by reading the graph.)

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2. (a) Plot and label these points on the graph:

$$\begin{aligned} P &= (2, d(2)) \\ Q_2 &= (4, d(4)) \\ Q_1 &= (3, d(3)) \\ Q_{0.5} &= (2.5, d(2.5)) \\ Q_{0.01} &= (2.01, d(2.01)) \end{aligned}$$

(b) Sketch the line through P and Q_2
Sketch the line through P and Q_1
Sketch the line through P and $Q_{0.5}$
Sketch the line through P and $Q_{0.01}$

(c) Compute the slopes of the lines in (b).

(d) We define the average velocity as follows: average velocity$=\frac{\text{distance traveled}}{\text{time elapsed}}$.
Explain why the slope of the line $\overline{PQ_2}=$ the average velocity from $t = 2$ seconds to $t = 4$ seconds.

(e) Find the average velocity over the following time intervals:

$t = 2$ seconds to $t = 4$ seconds:

$t = 2$ seconds to $t = 3$ seconds:

$t = 2$ seconds to $t = 2.5$ seconds:

$t = 2$ seconds to $t = 2.01$ seconds:

(f) The average velocities in (e) approach a number as the time interval gets smaller and smaller. Guess this number.

3. Let h be a small constant positive number and define $Q_h = (2+h, d(2+h))$. Compute the slope of the secant line connecting P and Q_h by simplifying:

$$slope = \frac{(\text{y coordinate}\, Q_h) - (\text{y coordinate}\, P)}{(\text{t coordinate}\, Q_h) - (\text{t coordinate}\, P)}$$

so there is no h in the denominator. This slope $=$ the average velocity on the time interval $t = 2$ seconds to $t = 2 + h$ seconds.

4. What number do you get when you plug $h = 0$ into the simplified expression in problem 3 above? This is called the instantaneous velocity at $t = 2$ seconds.

5. Draw a line through P with slope equal to the number computed in the previous step. How would you describe this line relative to the graph?

Worksheet

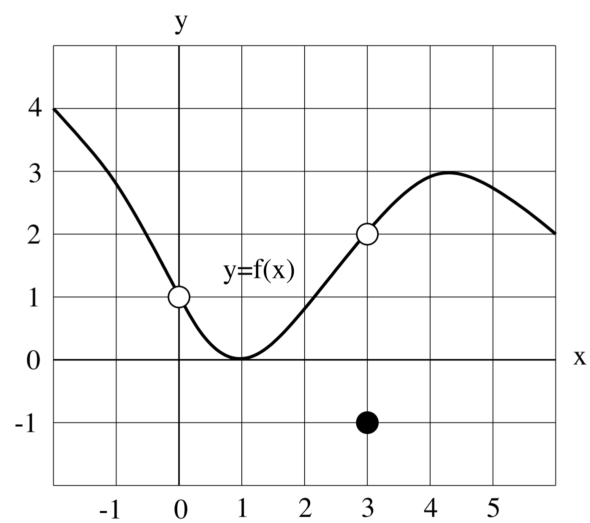
Continuity and Limits Math 124

Introduction

This worksheet discusses a method of computing limits for some special functions.

1. Use the graph of the function f at right to answer the following questions. Assume this is the entire graph of f.

   (a) What is the domain of f?



   (b) Find $\lim_{x\to 3} f(x)$.

   (c) Find $\lim_{x\to 0} f(x)$.

   (d) What is $f(3)$?

DEFINITION: A function g is called **continuous at** a if two things are true:

(i) a is a point in the domain of g and (ii) $\lim_{x\to a} g(x) = g(a)$.

A function is just called **continuous** if it is continuous at every point in its domain.

2. Answer the following questions for the function f whose graph is above. Explain your answers.

   (a) Is f continuous at 3?
   (b) Is f continuous at 0?
   (c) Is f continuous at 1?
   (d) Is f continuous?

Not every function is continuous, but many are. It's often important to know ahead of time that a function we want to work with is continuous, because that makes limit calculations easy. The following kinds of functions are *always* continuous:

- constant functions
- linear functions
- polynomials
- trigonometric functions
- exponential functions

There are many other kinds of continuous functions; some will be explored later in the course. You can also combine continuous functions to make new continuous functions. For example, you can add, multiply or compose continuous functions and be guaranteed to get a continuous function. Division, on the other hand, sometimes creates complications, as we'll see below.

3. Find $\lim_{x \to 2} 3x^2 - 2x + 4.$ Explain your solution.

4. Can you find $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$ by plugging in $x = 1$? Explain.

Sometimes we can find a limit of a function where it isn't continuous or defined by first simplifying the function, so that it resembles a continuous function.

5. Simplify the function in problem number 4, and find the limit of the resulting expression as $x \to 1$. Why can you say that this is the same as the limit you were asked to find in problem number 4?

6. Let $e(x) = 125x^2 - 6.25x^3$.

(a) Simplify the expression $\frac{e(5+h)-e(5)}{(5+h)-(5)}$ as much as possible; there should not be an h in the denominator of any fraction in the simplified expression.

(b) Is $\frac{e(5+h)-e(5)}{(5+h)-(5)}$ continuous at $h = 0$? Why?
Is the simplified expression in part (a) continuous at $h = 0$? Why?

(c) Calculate $\displaystyle\lim_{h \to 0} \frac{e(5+h) - e(5)}{(5+h) - (5)}$ using the approach outlined in problem 5.

7. Let $e(x) = 125x^2 - 6.25x^3$, as in problem 6. Let x be an unknown constant.

(a) Simplify the expression $\frac{e(x+h)-e(x)}{(x+h)-(x)}$ as much as possible; there should not be an h in the denominator of any fraction in the simplified expression.

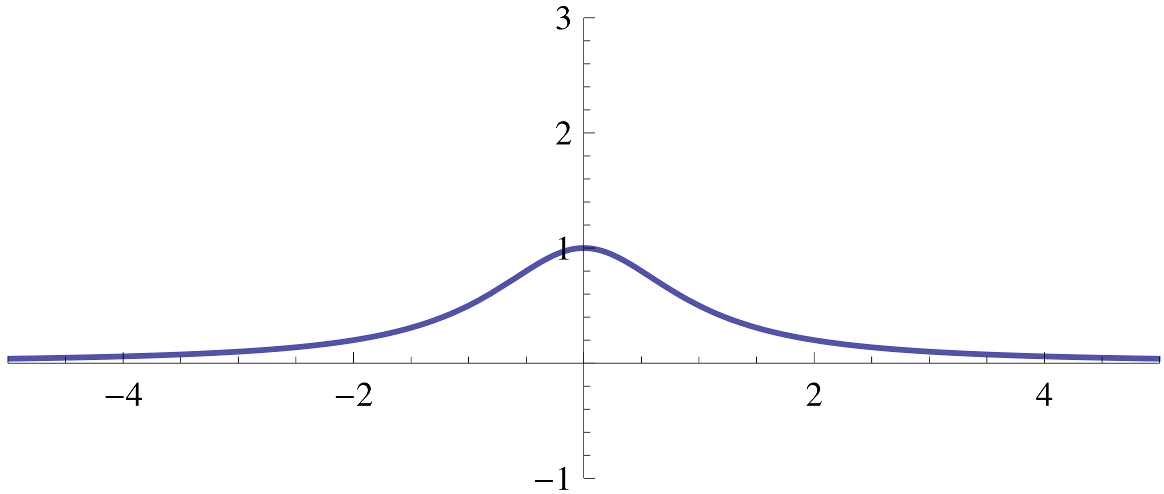
(b) Is $\frac{e(x+h)-e(x)}{(x+h)-(x)}$ continuous at $h = 0$? Why?
Is the simplified expression in part (a) continuous at $h = 0$? Why?

(c) Calculate $\displaystyle\lim_{h\to 0} \frac{e(x+h)-e(x)}{(x+h)-(x)}$ using the approach outlined in problem 5.

The Derivative Function | Math 124

Introduction

This worksheet will work with the function $y = f(x) = \dfrac{1}{x^2+1}$ whose graph is given below:



Recall that the derivative of $f(x)$ at $x = a$, denoted by $f'(a)$, is the instantaneous rate of change of $f(x)$ at $x = a$, which is the slope of the tangent line to the graph of $f(x)$ at the point $(a, f(a))$.

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1. Looking only at the graph of $y = f(x)$ above, answer these questions about $f'(a)$; you should be able to answer these questions without doing any calculuations:

   (a) For which a is $f'(a)$ positive?

   (b) For which a is $f'(a)$ negative?

   (c) For which a is $f'(a)$ zero?

   (d) What is $\lim_{a\to\infty} f'(a)$?

   (e) What is $\lim_{a\to-\infty} f'(a)$?

   (f) What is $\lim_{a\to 0} f'(a)$?

   (g) If you consider the slopes of all of the possible tangent lines to the graph of $y = f(x)$, is there a largest slope? If so, approximately where is this tangent line on the graph of $y = f(x)$? If not, why not.

   (h) If you consider the slopes of all of the possible tangent lines to the graph of $y = f(x)$, is there a smallest slope? If so, approximately where is this tangent line on the graph of $y = f(x)$? If not, why not.

2. Let h be a small real number, fix the point $P = (a, f(a))$ on the graph of $y = f(x)$ and the nearby point $Q = (a+h, f(a+h))$.

   (a) Let $s(h) =$ slope of the secant line through P and Q. Find a formula for $s(h)$. Simplify your answer so that it is a single rational expression in h and simplify it as much as possible.

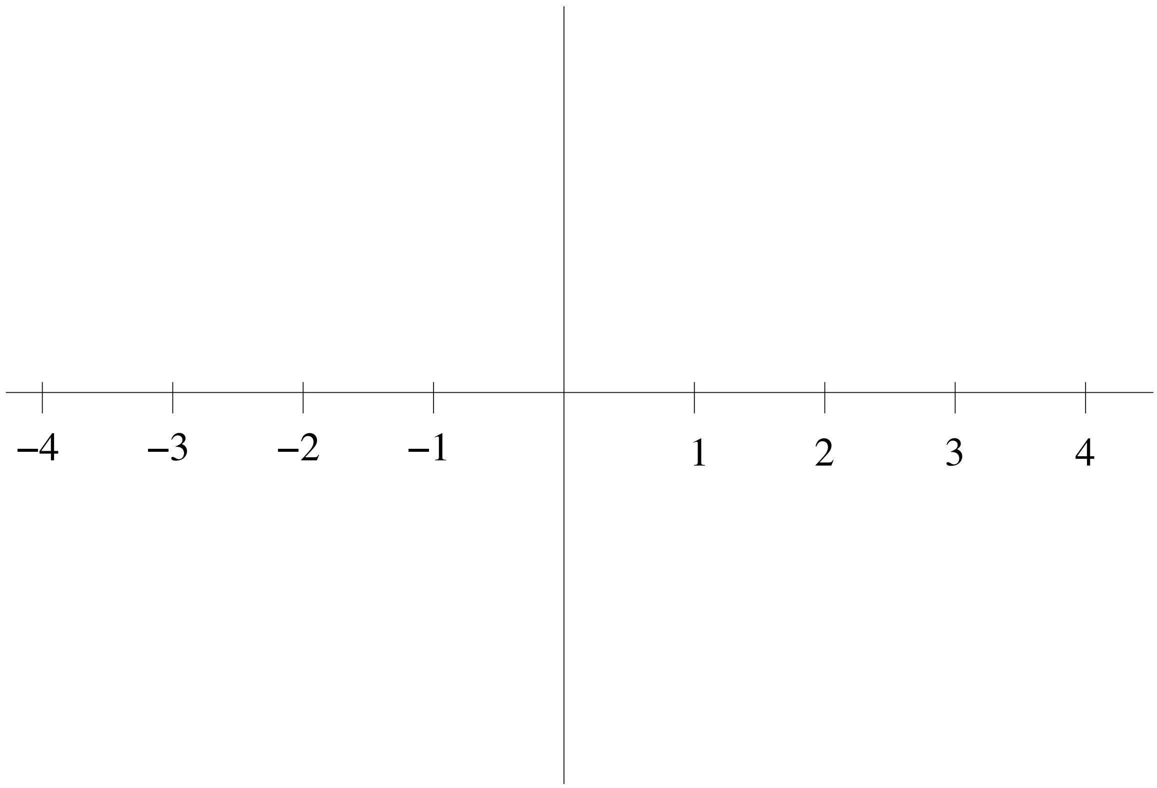
   (b) Calculate $f'(a) = \lim_{h \to 0} s(h)$.

   (c) Create a table of values:

a	$f'(a)$
-4	
-3	
-2	
-1	
-.9	
-.8	
-.7	
-.6	
-.5	
-.4	
-.3	
-.2	
-.1	
0	

a	$f'(a)$
.1	
.2	
.3	
.4	
.5	
.6	
.7	
.8	
.9	
1	
2	
3	
4	

(d) Plot the points $(a, f'(a))$ from your table in the previous step below and "connect the dots". Replacing "a" by "x" in 2(b), the sketch below is the graph of $y = f'(x) = \frac{-2x}{(1+x^2)^2}$; the *derivative function* $y = f'(x)$. Discuss how your answers in question 1 relate to this picture.



3. Starting with the function $y = g(x) = \dfrac{-2x}{(1+x^2)^2}$ and a real number $x = a$, follow the steps in 2(a),(b) to calculate the formula for $g'(a)$.

4. How can you use the work in 3. to exactly determine the locations on the graph of $y = f(x)$ where the tangent line has largest and smallest slope (i.e. the exact answers to 1(g),(h))?

Worksheet-Week #6

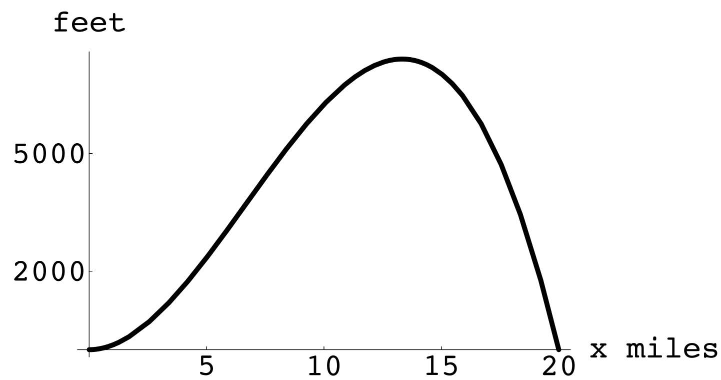
The Hiker and the Submarine Math 124

Introduction

This worksheets introduces the problem of find maximum and minimum values of functions using techniques of calculus.

1. You are planning a 20 mile hike in the Cascades. The guidebook provides you with a chart, plotting elevation above the trailhead (in feet) as a function of the distance hiked (in miles). This plot is shown and is modeled by the function

$$e(x) = 125x^2 - 6.25x^3.$$



(a) Using the graph of $e(x)$, explain in words how the tangent lines to the graph relate to the difficulty of the hike.

(b) What can you say about the slope of the tangent line at the point of highest elevation on the graph?

(c) Find the instantaneous rate of change of elevation as a function of $x =$ (distance hiked). Include units in your answer.

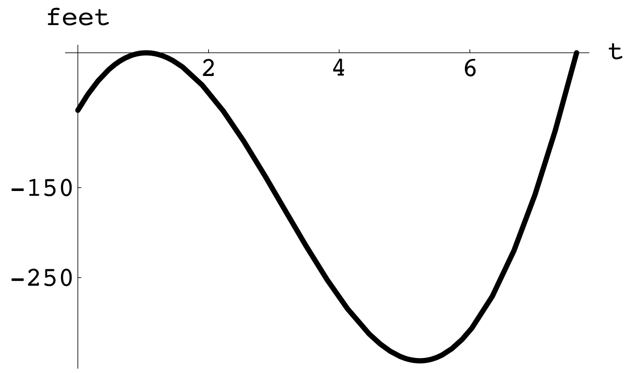
(d) Determine how far the hiker will have traveled when he reaches the highest point on the trail.

(e) What is the elevation of the highest point on the trail? Give your answer accurate to the nearest whole foot.

2. A submarine executes a diving drill. Starting from 64 feet below sea level, the submarine surfaces, then dives several hundred feet before resurfacing. The altitude of the submarine in feet is given by the function

$$a(t) = 500\cos\left(\frac{t}{2}\right) + 125t - 564,$$

where t represents the number of minutes that have passed since the beginning of the drill. The graph of a is plotted at right.



(a) What can you say about the tangent line to the graph at the lowest point on the dive?

(b) After how many minutes will the submarine reach the lowest point? Round your answer to two decimal places.

(c) What is the greatest depth the submarine will reach on this dive? Round your answer to the nearest foot.

(d) What is the greatest rate at which the submarine's depth will increase during the dive? Include units in your answer.

Worksheet-Week #7

Sinusoidal functions and the Piston Problem   Math 124

Introduction. In this worksheet, we will begin by investigating derivatives of sinusoidal functions. We then model the motion of a piston and study its velocity and acceleration.

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1. **Derivatives of Sinusoidal Functions** Let $f(t) = 4\sin(\pi t)$. Calculate each derivative and put it into standard sinusoidal form. You will need to use the identity: $\cos(\theta) = \sin(\theta + \frac{\pi}{2})$.

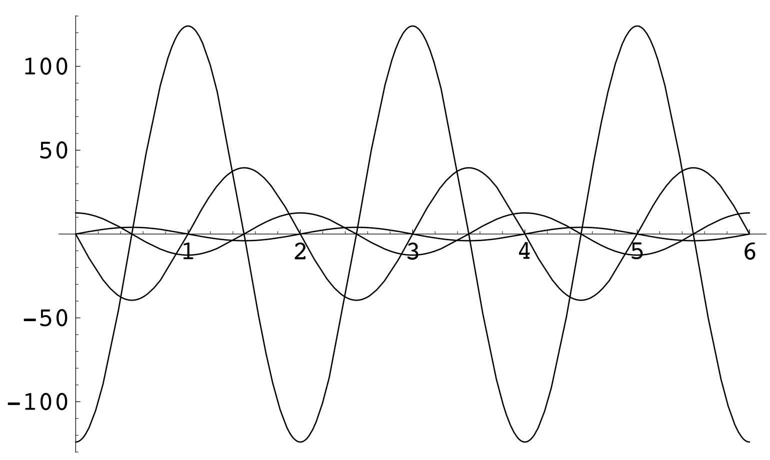
   (a) $f'(t) =$

   (b) $f''(t) =$

   (c) $f'''(t) =$

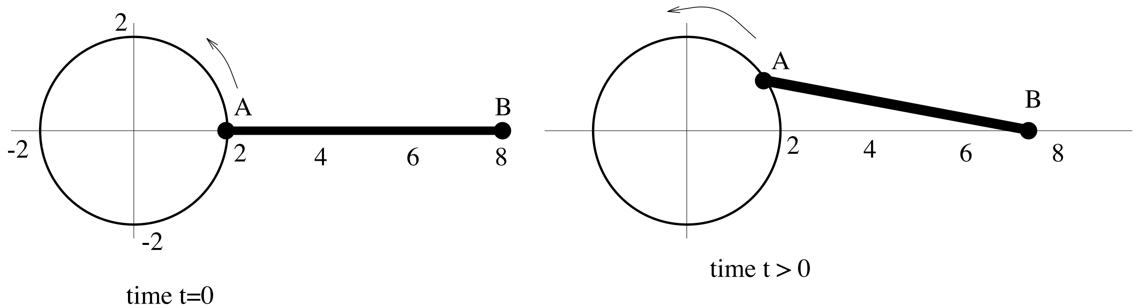
   (d) $f^{(4)}(t) =$

   (e) Below are the graphs of $f(t)$ and the first three derivatives; identify each curve.



   (f) True or False: The derivative of a sinuosidal function is a sinusodial function.

The Piston A six foot long rod is attached at one end A to a point on a wheel of radius 2 feet, centered at the origin. The other end B is free to move back and forth along the x-axis. The point A is at $(2,0)$ at time $t=0$, and the wheel rotates counterclockwise at constant speed with an angular speed of 3/2 revolutions per minute.



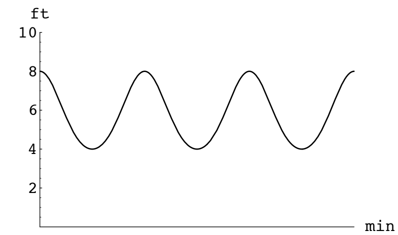
2. Let $x(t)$ be the x-coordinate of the point B as a function of time t minutes; what is the formula for $x(t)$?

3. Calculate $x'(t)$.

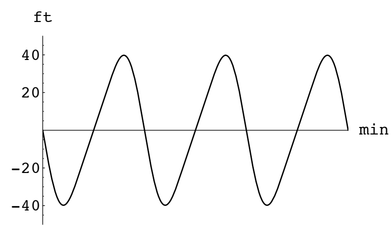
5. **Use calculus** to determine when the velocity of B is zero. What is the picture of the piston when this happens?

4. The graphs of $x(t)$, $x'(t)$ and $x''(t)$ on the domain $0 \leq t \leq 1$ minute are below. The graph of $x(t)$ looks like the graph of a sinusoidal function. Is $x(t)$ a sinusoidal function? Is $x(t)$ a periodic function? How can you tell? (Hint: Go back to 1.)

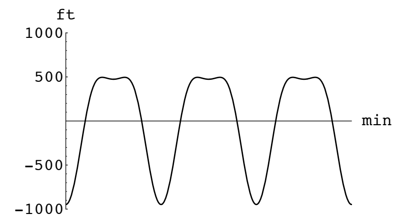
$x(t)$



$x'(t)$



$x''(t)$



Worksheet- #10

Brightness of a star — Math 124

Introduction

Suppose an astronomer models the brightness of a star by the function $B(t) = ae^{1/t}$ where t is time measured in years and a is a positive constant. Her model applies to the time interval $(-\infty, 0)$, assuming that $t = 0$ is the present moment. (Note: This problem appeared on the final exam, winter 2002.)

1. Find $\lim_{t \to -\infty} B(t)$. (Do not use L'Hopitals Rule.)

2. In this model, what happens to the brightness of the star as $t \to 0^-$? Show your computation.

3. Find the rate of change of the brightness of the star for $-\infty < t < 0$.

4. Compute $B''(t)$ and use it to determine the intervals in which the *rate of change* of brightness of the star is (a) decreasing (b) increasing.

5. Sketch the graph of $B(t)$ for $-\infty < t < 0$. Make sure to indicate the coordinates of any points of interest on the graph, such as local extrema or inflection points (if there are any).

If your group has finished the worksheet, work on this problem:

Brooke is located 5 miles out from the nearest point A along a straight shoreline in her seakayak. Hunger strikes and she wants to make it to *Kono's* for lunch; see picture. Brooke can paddle 2 mph and walk 4 mph. If she paddles along a straight line course to the shore, find an equation that computes the the total time to reach lunch in terms of the location where Brooke beaches the boat. Where should she beach the kayak to eat as soon as possible?