More examples with lines and planes

If two planes are not parallel, they will intersect, and their intersection will be a line. Given the equations of two non-parallel planes, we should be able to determine that line of intersection. Here is an example of doing just that.

Example: Suppose we want to find the intersection of the planes

$$P_1 : 3x - y + 6z = 5$$

and

$$P_2 : 2x + 3y - z = 10$$

Note that their normal vectors, $< 3, -1, 6 >$ and $< 2, 3, -1 >$ are not parallel (how can you tell?) so the planes are not parallel.

The key idea to finding the line of intersection is this: since the line lies in both planes, its direction vector is orthogonal to both planes' normal vectors.

Let L be the line of intersection. Then its direction vector is orthogonal to $< 3, -1, 6 >$ and $< 2, 3, -1 >$, and hence the direction vector is parallel to the cross product

$$< 3, -1, 6 > \times < 2, 3, -1 >=< -17, 15, 11 > \text{ (check!)}$$

So, we can take $< -17, 15, 11 >$ as the direction vector of L, and now we just need a point on the line.

Unless a line is parallel to the yz-plane, it will hit the yz-plane. Our line L is not parallel to the yz-plane (how do we know?), L hits the yz-plane; hence, there is a point on L at which $x = 0$, and, from the plane equations,

$$-y + 6z = 5$$

and

$$3y - z = 10$$

Those equations together give

$$y = \frac{65}{17} \quad \text{and} \quad z = \frac{25}{17}. \text{ (check!)}$$

Hence, the point $(0, 65/17, 25/17)$ is on L, so L has equations

$$x = -17t, y = \frac{65}{17} + 15t, z = \frac{25}{17} + 11t$$

Now, let's find the equation of the plane containing a given line, and a given point.

Example: Suppose we want the equation of the plane containing the line L from the last example, and the point $(5, 4, 3)$. If we had two vectors in the plane, then we could find the cross product, and use that as the normal to the plane. We already know $< -17, 15, 11 >$ is in the plane, and we know two points in the plane, so

$$\left\langle 5-0, 4-\frac{65}{17}, 3-\frac{25}{17} \right\rangle = \left\langle 5, \frac{3}{17}, \frac{26}{17} \right\rangle$$

is also in the plane.

The cross product gives us

$$< -17, 15, 11 > \times \left\langle 5, \frac{3}{17}, \frac{26}{17} \right\rangle =< 21, 81, -78 > \text{ (check!)}$$

so the plane has equation

$$21(x-5) + 81(y-4) - 78(z-3) = 0$$

which can be rewritten as

$$21x + 81y - 78z - 195 = 0$$

or

$$21x + 81y - 78z = 195.$$