MATH 126
Exam I Review - Solutions

1. $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{2 \cdot 2!}\left(x - \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{2 \cdot 3!}\left(x - \frac{\pi}{4}\right)^3 + ...$

2. We know that the Taylor series for $g(x)$ has the form
$$\sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} x^k.$$
So, $\frac{g^{(3)}(0)}{3!}$ is the coefficient on the x^3 term of the series given. That is,
$$\frac{g^{(3)}(0)}{3!} = -25.$$
So, $g^{(3)}(0) = -25 \cdot 3! = -150$.

3. (a) $T_2(x) = x + x^2$
   (b) We may use the fact that
$$|f'''(x)| = |2e^x(\cos x - \sin x)| \leq 4e^x \leq 4e^{0.1}$$
   on the interval $-0.1 \leq x \leq 0.1$, so the error is no more than
$$\frac{4e^{0.1}}{3!}|0.1|^3 = 0.000736781.....$$

4. Since
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
for all x, we have
$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$
and
$$xe^{x^2} = x + x^3 + \frac{x^5}{2!} + \frac{x^7}{3!} + \cdots$$
Also,
$$\frac{1}{4 + x^2} = \left(\frac{1}{4}\right)\frac{1}{1 - \left(-\left(\frac{x}{2}\right)^2\right)} = \frac{1}{4}\sum_{k=0}^{\infty}\left(-\left(\frac{x}{2}\right)^2\right)^k = \frac{1}{4} - \frac{x^2}{4 \cdot 4} + \frac{x^4}{4 \cdot 2^4} - \frac{x^6}{4 \cdot 2^6} + \cdots$$
Combining these results yields
$$f(x) = -\frac{1}{4} + x + \frac{x^2}{4 \cdot 4} + x^3 + \cdots.$$

5. (a) $T_2(x) = -2 + 3(x-1) + 14(x-1)^2$
   (b) error ≤ 0.09375.

6. $\dfrac{1}{4!}$.

7. It is necessary to find the first two derivatives of $f(x)$:

$$f'(x) = \frac{1}{x \ln x}$$

$$f''(x) = -\frac{1 + \ln x}{(x \ln x)^2}$$

Evaluating f, f', and f'' at $x = e$ gives the coefficients of $T_2(x)$ for $f(x)$:

$$T_2(x) = \frac{1}{e}(x-e) - \frac{1}{e^2}(x-e)^2$$

8. The first four non-zero terms of the Taylor series for $\sin x$ are

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

so the first four non-zero terms of the Taylor series for $\sin x^2$ are

$$x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!}$$

Integrating this from 0 to 2 gives

$$\left(\frac{1}{3}x^3 - \frac{1}{7 \cdot 3!}x^7 + \frac{1}{11 \cdot 5!}x^{11} - \frac{1}{15 \cdot 7!}x^{15}\right)\Bigg|_0^2 = \frac{8}{3} - \frac{128}{42} + \frac{2048}{1320} - \frac{32768}{75600} = 0.7371236.$$

9. We use

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ...$$

so that

$$\frac{1}{1-(-5x)} = 1 - 5x + (5x)^2 - (5x)^3 + ...$$

and

$$\frac{1}{3+x} = \left(\frac{1}{3}\right)\frac{1}{1-(-\frac{1}{3}x)} = \frac{1}{3}\left(1 - \frac{1}{3}x + \left(\frac{1}{3}x\right)^2 - \left(\frac{1}{3}x\right)^3 + ...\right) = \frac{1}{3} - \frac{1}{9}x + \frac{1}{27}x^2 - \frac{1}{81}x^3 + ...$$

Adding these together, we have

$$\frac{1}{1+5x} + \frac{1}{3+x} = \frac{4}{3} - \frac{46}{9}x + \frac{676}{27}x^2 - \frac{10126}{81}x^3 + ...$$

10.

$$\frac{dx}{dt} = \frac{2}{\sqrt{t}} \text{ and } \frac{dy}{dt} = t^2 - \frac{1}{t^3}$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\frac{4}{t} + t^4 - \frac{2}{t} + \frac{1}{t^6}} = \sqrt{t^4 + \frac{2}{t} + \frac{1}{t^6}} = \sqrt{\left(t^2 + \frac{1}{t^3}\right)^2} = t^2 + \frac{1}{t^3}.$$

The length of the curve is:

$$L = \int_1^4 t^2 + \frac{1}{t^3}\, dt = \frac{1}{3}t^3 - \frac{1}{2t^2}\Big|_1^4 = ...$$

11. No. Basically you can show that if there was an intersection, it would occur when $\sin\theta = 1$. However, there is no point on the curve where $\sin\theta = 1$ (why not?), so there is no intersection.

12. The intersection will occur when $r = \frac{1}{3}$, i.e., when $\theta = 3$. The slope of the tangent line is

$$\frac{dy}{dx} = \frac{-\frac{1}{9}\sin 3 + \frac{1}{3}\cos 3}{-\frac{1}{9}\cos 3 - \frac{1}{3}\sin 3}$$

13. Since $y = r\sin\theta$, $\csc\theta = \frac{r}{y}$. Note that the curve $r = 4\csc\theta$ does not go through the origin since $\csc\theta$ is never equal to 0. This means that r is never 0. So,

$$r = 4\csc\theta \Rightarrow r = \frac{4r}{y} \Rightarrow y = 4.$$

That is, the polar curve $r = 4\csc\theta$ is the horizontal line $y = 4$ in the Cartesian plane. The intersection of this line with the line $y = x$ is the point with Cartesian coordinates $(4, 4)$ and polar coordinates $(4\sqrt{2}, \frac{\pi}{4})$.

14. We need all x such that the dot product of the two vectors is equal to 0:

$$\langle 4, 5, x\rangle \cdot \langle 3x, 7, x\rangle = 12x + 35 + x^2 = (x + 7)(x + 5).$$

This is 0 for $x = -7$ and $x = -5$.

15. Many (in fact, infinitely many) correct answers to this one. Here is one. $\langle 0, \frac{35}{\sqrt{34}}, \frac{21}{\sqrt{34}}\rangle$.

16.

$$\theta = \cos^{-1}\left(\frac{4}{\sqrt{26}\sqrt{11}}\right) = 1.80958408790828020.$$

17. Since the vectors are orthogonal,

$$< x, 3, 2 > \cdot < 2, 3, x >= 0$$

so

$$2x + 9 + 2x = 0$$

from which we conclude that $x = -\frac{9}{4}$.

18. Suppose $P(x, y, z)$ is a point on the sphere. Let A be the point $(5, 5, 5)$ and O be the origin. Then the distance from P to O is twice the distance from P to A:

$$\sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = 2\sqrt{(x-5)^2 + (y-5)^2 + (z-5)^2}.$$

Square both sides and complete the square(s) to get the standard form of the equation for the sphere. Read off the center and radius.

ANSWER: center $\left(\frac{20}{3}, \frac{20}{3}, \frac{20}{3}\right)$, radius $\frac{10}{\sqrt{3}}$.

19. The slope of the tangent line to the curve is

$$\frac{dy}{dx} = \frac{2(t+1)}{3(t^2-1)}.$$

The slope of the line, $x = 3t + 5$, $y = t - 6$, is

$$\frac{dy}{dx} = \frac{1}{3}.$$

Set these two slopes equal to each other and solve for t: $t = 3$. The point on the curve that corresponds to $t = 3$ is $(18, 15)$.

20. There are many different ways to solve this. Here's one:

We know

$$\sin^2 t + \cos^2 t = 1$$

and

$$\cos t = x$$

and

$$\sin t = y + \cos t = y + x$$

so that

$$(y + x)^2 + x^2 = 1$$

and we're done.

Here's another way: notice that we can write

$$\sin t = \pm\sqrt{1 - \cos^2 t} = \pm\sqrt{1 - x^2}$$

so that

$$y = \pm\sqrt{1 - x^2} - x.$$

21. If $t = e$ then $x = 0$, and $y = 0$. We have

$$\frac{dx}{dt} = \frac{1}{t \ln t} = \frac{1}{e}$$

when $t = e$, and

$$\frac{dy}{dt} = \frac{1}{t} - 2(\ln t)\frac{1}{t} = \frac{1}{e} - \frac{2}{e} = -\frac{1}{e}$$

when $t = e$.

Thus, when $t = e$,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -1$$

so the tangent line has equation $y = -x$.