MATH 126
Exam I Review - Solutions

1. \[ \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \left( x - \frac{\pi}{4} \right) - \frac{\sqrt{2}}{2 \cdot 2!} \left( x - \frac{\pi}{4} \right)^2 - \frac{\sqrt{2}}{2 \cdot 3!} \left( x - \frac{\pi}{4} \right)^3 + \ldots \]

2. We know that the Taylor series for \( g(x) \) has the form
   \[ \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} x^k. \]
   So, \( \frac{g^{(3)}(0)}{3!} \) is the coefficient on the \( x^3 \) term of the series given. That is,
   \[ \frac{g^{(3)}(0)}{3!} = -25. \]
   So, \( g^{(3)}(0) = -25 \cdot 3! = -150. \)

3. (a) \( T_2(x) = x + x^2 \)
   (b) We may use the fact that
   \[ |f'''(x)| = |2e^x(\cos x - \sin x)| \leq 4e^x \leq 4e^{0.1} \]
   on the interval \(-0.1 \leq x \leq 0.1\), so the error is no more than
   \[ \frac{4e^{0.1}}{3!} |0.1|^3 = 0.000736781\ldots. \]

4. Since
   \[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \]
   for all \( x \), we have
   \[ e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \ldots \]
   and
   \[ xe^{x^2} = x + x^3 + \frac{x^5}{2!} + \frac{x^7}{3!} + \ldots \]
   Also,
   \[ \frac{1}{4 + x^2} = \left( \frac{1}{4} \right) \frac{1}{1 - \left( \frac{x}{2} \right)^2} = \frac{1}{4} \sum_{k=0}^{\infty} \left( \frac{x}{2} \right)^{2k} = \frac{1}{4} - \frac{x^2}{4 \cdot 4} + \frac{x^4}{4 \cdot 2^4} - \frac{x^6}{4 \cdot 2^6} + \ldots \]
   Combining these results yields
   \[ f(x) = -\frac{1}{4} + x + \frac{x^2}{4 \cdot 4} + x^3 + \ldots. \]
5. (a) \( T_2(x) = -2 + 3(x - 1) + 14(x - 1)^2 \)
(b) error \( \leq 0.09375 \).

6. \( \frac{1}{4!} \).

7. It is necessary to find the first two derivatives of \( f(x) \):
   
   \[
   f'(x) = \frac{1}{x \ln x}
   \]
   \[
   f''(x) = -\frac{1 + \ln x}{(x \ln x)^2}
   \]

   Evaluating \( f, f', \) and \( f'' \) at \( x = e \) gives the coefficients of \( T_2(x) \) for \( f(x) \):
   
   \[
   T_2(x) = \frac{1}{e}(x - e) - \frac{1}{e^2}(x - e)^2
   \]

8. The first four non-zero terms of the Taylor series for \( \sin x \) are
   
   \[
   x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}
   \]

   so the first four non-zero terms of the Taylor series for \( \sin x^2 \) are
   
   \[
   x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!}
   \]

   Integrating this from 0 to 2 gives
   
   \[
   \left( \frac{1}{3}x^3 - \frac{1}{7 \cdot 3!}x^7 + \frac{1}{11 \cdot 5!}x^{11} - \frac{1}{15 \cdot 7!}x^{15} \right)^2 \bigg|_0^2 = \frac{8}{3} - \frac{128}{42} + \frac{2048}{1320} - \frac{32768}{75600} = 0.7371236.
   \]

9. We use
   
   \[
   \frac{1}{1 - x} = 1 + x + x^2 + x^3 + \ldots
   \]

   so that
   
   \[
   \frac{1}{1 - (-5x)} = 1 - 5x + (5x)^2 - (5x)^3 + \ldots
   \]

   and
   
   \[
   \frac{1}{3 + x} = \frac{1}{3} \left( \frac{1}{1 - (-\frac{1}{3}x)} \right) = \frac{1}{3} \left( 1 - \frac{1}{3}x + \left( \frac{1}{3}x \right)^2 - \left( \frac{1}{3}x \right)^3 + \ldots \right) = \frac{1}{3} - \frac{1}{9}x + \frac{1}{27}x^2 - \frac{1}{81}x^3 + \ldots
   \]

   Adding these together, we have
   
   \[
   \frac{1}{1 + 5x} + \frac{1}{3 + x} = \frac{4}{3} - \frac{46}{9}x + \frac{676}{27}x^2 - \frac{10126}{81}x^3 + \ldots
   \]
10. \[
\frac{dx}{dt} = 2 \sqrt{t} \text{ and } \frac{dy}{dt} = t^2 - \frac{1}{t^3}
\]
\[
\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\frac{4}{t} + t^4 - \frac{2}{t} + \frac{1}{t^6}} = \sqrt{t^4 + \frac{2}{t} + \frac{1}{t^6}} = \sqrt{\left(t^2 + \frac{1}{t^3}\right)^2} = t^2 + \frac{1}{t^3}.
\]

The length of the curve is:
\[
L = \int_{1}^{4} t^2 + \frac{1}{t^3} \ dt = \left. \frac{1}{3} t^3 - \frac{1}{2} t^{\frac{2}{3}} \right|_{1}^{4} = ...\]

11. No. Basically you can show that if there was an intersection, it would occur when \(\sin \theta = 1\). However, there is no point on the curve where \(\sin \theta = 1\) (why not?), so there is no intersection.

12. The intersection will occur when \(r = \frac{1}{3}\), i.e., when \(\theta = 3\). The slope of the tangent line is
\[
\frac{dy}{dx} = -\frac{1}{9} \sin 3 + \frac{1}{3} \cos 3
\]
\[= \frac{-1}{9} \cos 3 - \frac{1}{3} \sin 3
\]

13. Since \(y = r \sin \theta\), \(\csc \theta = \frac{r}{y}\). Note that the curve \(r = 4 \csc \theta\) does not go through the origin since \(\csc \theta\) is never equal to 0. This means that \(r\) is never 0. So,

\(r = 4 \csc \theta \Rightarrow r = \frac{4r}{y} \Rightarrow y = 4\).

That is, the polar curve \(r = 4 \csc \theta\) is the horizontal line \(y = 4\) in the Cartesian plane. The intersection of this line with the line \(y = x\) is the point with Cartesian coordinates \((4, 4)\) and polar coordinates \((4\sqrt{2}, \frac{\pi}{4})\).

14. We need all \(x\) such that the dot product of the two vectors is equal to 0:
\[
\langle 4, 5, x \rangle \cdot \langle 3x, 7, x \rangle = 12x + 35 + x^2 = (x + 7)(x + 5).
\]

This is 0 for \(x = -7\) and \(x = -5\).

15. Many (in fact, infinitely many) correct answers to this one. Here is one.
\[\langle 0, \frac{35}{\sqrt{34}}, \frac{21}{\sqrt{34}} \rangle.\]

16. \[
\theta = \cos^{-1} \left( \frac{4}{\sqrt{26\sqrt{11}}} \right) = 1.80958408790828020.
\]

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17. Since the vectors are orthogonal,
\[ \langle x,3,2 \rangle \cdot \langle 2,3,x \rangle = 0 \]
so
\[ 2x + 9 + 2x = 0 \]
from which we conclude that \( x = -\frac{9}{4} \).

18. Suppose \( P(x,y,z) \) is a point on the sphere. Let \( A \) be the point \((5,5,5)\) and \( O \) be the origin. Then the distance from \( P \) to \( O \) is twice the distance from \( P \) to \( A \):
\[ \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = 2\sqrt{(x-5)^2 + (y-5)^2 + (z-5)^2}. \]
Square both sides and complete the square(s) to get the standard form of the equation for the sphere. Read off the center and radius.
\[ \text{ANSWER: center } \left( \frac{20}{3}, \frac{20}{3}, \frac{20}{3} \right), \text{ radius } \frac{10}{\sqrt{3}}. \]

19. The slope of the tangent line to the curve is
\[ \frac{dy}{dx} = \frac{2(t+1)}{3(t^2-1)}. \]
The slope of the line, \( x = 3t + 5, y = t - 6 \), is
\[ \frac{dy}{dx} = \frac{1}{3}. \]
Set these two slopes equal to each other and solve for \( t \): \( t = 3 \). The point on the curve that corresponds to \( t = 3 \) is \((18,15)\).

20. There are many different ways to solve this. Here’s one:
We know
\[ \sin^2 t + \cos^2 t = 1 \]
and
\[ \cos t = x \]
and
\[ \sin t = y + \cos t = y + x \]
so that
\[ (y + x)^2 + x^2 = 1 \]
and we’re done.
Here’s another way: notice that we can write
\[ \sin t = \pm \sqrt{1 - \cos^2 t} = \pm \sqrt{1 - x^2} \]
so that
\[ y = \pm \sqrt{1 - x^2} - x. \]
21. If \( t = e \) then \( x = 0 \), and \( y = 0 \). We have

\[
\frac{dx}{dt} = \frac{1}{t \ln t} = \frac{1}{e}
\]

when \( t = e \), and

\[
\frac{dy}{dt} = \frac{1}{t} - 2 (\ln t) \frac{1}{t} = \frac{1}{e} - \frac{2}{e} = -\frac{1}{e}
\]

when \( t = e \).

Thus, when \( t = e \),

\[
\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -1
\]

so the tangent line has equation \( y = -x \).