Euler’s Formula via Taylor Series Worksheet

In this worksheet, you will prove the formula

\[ e^{i\theta} = \cos \theta + i \sin \theta. \]

This is perhaps the most famous of all formulas in mathematics. It is known as Euler’s formula, for Leonard Euler, a Swiss mathematician who lived from 1707 to 1783. The formula was actually first proved by Roger Cotes in 1714, an English mathematician who lived from 1682 to 1716.

The \( i \) in the formula is known as the imaginary unit. It is defined to be the non-real number with the property that

\[ i^2 = -1. \]

Since \( i \) is not a real number, it is said to be imaginary, and it gives rise to the set of complex numbers. Complex numbers are numbers of the form

\[ a + bi \]

where \( a \) and \( b \) are real numbers. Euler’s formula expresses an equality between two ways of representing a complex number.

You can use Taylor series to prove the formula.

Here are a few steps.

1. The first thing to do is to check out what happens to powers of \( i \). Since

\[ i^2 = -1, \]

we have \( i^3 = -i \). What is \( i^4 \)? \( i^5 \)? \( i^{62} \)? What is \( i^n \) for a general positive integer \( n \)?

2. The Taylor series for \( e^x \),

\[ e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \]

converges for all real \( x \). In fact, for any complex number \( x \), the series converges to \( e^x \).

Use the last step to write out the first 8 (or more) terms of the series for \( e^{i\theta} \).

3. How does this compare to the Taylor series for \( \cos x \) and \( \sin x \)? Show how this gets us Euler’s formula.
This is a bit of a casual proof. By getting a general expression for the \( n \)-th term of the series for \( e^{i\theta} \), and our knowledge of the \( n \)-th term of the series for \( \cos \theta \) and \( \sin \theta \), the proof could be made completely solid.

**What can you do with Euler’s formula?**

1. If you let \( \theta = \pi \), Euler’s formula simplifies to what many claim is the most beautiful equation in all of mathematics. It does tie together three important constants, \( e \), \( i \), and \( \pi \) rather nicely.

2. We can get quick proofs for some trig identities from Euler’s formula. We need this fact: if \( a, b, c, \) and \( d \) are real numbers, and

   \[ a + bi = c + di \]

   then \( a = c \) and \( b = d \). That is, if two complex numbers are equal, then their real parts are equal and their imaginary parts are equal.

   Now, replacing \( \theta \) by \( n\theta \) in Euler’s formula we have

   \[ e^{in\theta} = \cos(n\theta) + i \sin(n\theta) \]

   However, the left side can be written as

   \[ e^{in\theta} = (e^{i\theta})^n = (\cos \theta + i \sin \theta)^n \]

3. Let \( n = 2 \) and expand to prove the two double-angle formulas

   \[ \cos 2\theta = \cos^2 \theta - \sin^2 \theta \]

   and

   \[ \sin 2\theta = 2 \sin \theta \cos \theta \]

4. Let \( n = 3 \) and expand to prove the less common triple-angle formulas

   \[ \cos 3\theta = \cos \theta \left( \cos^2 \theta - 3 \sin^2 \theta \right) = \cos \theta \left( 4 \cos^2 \theta - 3 \right) \]

   and

   \[ \sin 3\theta = \sin \theta \left( 3 \cos^2 \theta - \sin^2 \theta \right) = \sin \theta \left( 3 - 4 \sin^2 \theta \right) \]

You can see that, by letting \( n \) be other integers, many other formulas are possible.