# Euler's Formula via Taylor Series Worksheet

In this worksheet, you will prove the formula

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

This is perhaps the most famous of all formulas in mathematics. It is known as **Euler's formula**, for Leonard Euler, a Swiss mathematician who lived from 1707 to 1783. The formula was actually first proved by Roger Cotes in 1714, an English mathematician who lived from 1682 to 1716.

The $i$ in the formula is known as the **imaginary unit**. It is defined to be the non-real number with the property that

$$i^2 = -1.$$

Since $i$ is not a real number, it is said to be **imaginary**, and it gives rise to the set of **complex numbers**. Complex numbers are numbers of the form

$$a + bi$$

where $a$ and $b$ are real numbers. Euler's formula expresses an equality between two ways of representing a complex number.

You can use Taylor series to prove the formula.

Here are a few steps.

1. The first thing to do is to check out what happens to powers of $i$. Since

   $$i^2 = -1,$$

   we have $i^3 = -i$. What is $i^4$? $i^5$? $i^{62}$? What is $i^n$ for a general positive integer $n$?

2. The taylor series for $e^x$,

   $$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

   converges for all real $x$. In fact, for any *complex* number $x$, the series converges to $e^x$.

   Use the last step to write out the first 8 (or more) terms of the series for

   $$e^{i\theta}.$$

3. How does this compare to the Taylor series for $\cos x$ and $\sin x$? Show how this gets us Euler's formula.

This is a bit of a casual proof. By getting a general expression for the $n$-th term of the series for $e^{i\theta}$, and our knowledge of the $n$-th term of the series for $\cos\theta$ and $\sin\theta$, the proof could be made completely solid.

**What can you do with Euler's formula?**

1. If you let $\theta = \pi$, Euler's formula simplifies to what many claim is the most beautiful equation in all of mathematics. It does tie together three important constants, $e$, $i$, and $\pi$ rather nicely.

2. We can get quick proofs for some trig identities from Euler's formula. We need this fact: if $a, b, c,$ and $d$ are real numbers, and

$$a + bi = c + di$$

then $a = c$ and $b = d$. That is, if two complex numbers are equal, then their real parts are equal and their imaginary parts are equal.

Now, replacing $\theta$ by $n\theta$ in Euler's formula we have

$$e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$$

However, the left side can be written as

$$e^{in\theta} = \left(e^{i\theta}\right)^n = (\cos\theta + i\sin\theta)^n$$

3. Let $n = 2$ and expand to prove the two double-angle formulas

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

and

$$\sin 2\theta = 2\sin\theta\cos\theta$$

4. Let $n = 3$ and expand to prove the less common triple-angle formulas

$$\cos 3\theta = \cos\theta\left(\cos^2\theta - 3\sin^2\theta\right) = \cos\theta\left(4\cos^2\theta - 3\right)$$

and

$$\sin 3\theta = \sin\theta\left(3\cos^2\theta - \sin^2\theta\right) = \sin\theta\left(3 - 4\sin^2\theta\right)$$

You can see that, by letting $n$ be other integers, many other formulas are possible.