Worksheet on Curvature Formulas

For a curve with position function $\vec{r}'(t)$ we have the formula

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

for the curvature at the point given by $\vec{r}(t)$. In this worksheet, you will develop two practical formulas for finding the curvature of a plane curve.

1. Suppose $\vec{r}(t)$ defines a curve in the xy-plane. Then

$$\vec{r}(t) = \langle f(t), g(t), 0 \rangle$$

   for some real-valued functions f and g. Find $|\vec{r}' \times \vec{r}''|$.

2. Use your last calculation to show that

$$\kappa = \frac{|f'g'' - f''g'|}{(f'^2 + g'^2)^{3/2}}.$$

3. Use this formula to find the curvature of the archimedean spiral

$$x = t\cos t, y = t\sin t$$

   at the origin. Is the radius of curvature what you expected? Show that the radius of curvature is asymptotic to t as t goes to infinity.

4. Now, consider a plane curve defined by $y = f(x)$. Give a vector function $\vec{r}(t)$ that defines this curve.

5. Show that the curvature at the point (x, y) on this curve is

$$\kappa = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}$$

6. What does this last formula suggest about the curvature at points of inflection?

7. Consider the curve $y = x^3$. At what points is the curvature maximal?