How many local extrema and inflection points can a rational function have?

Here we treat only the case of rational functions of the form

$$f(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomials of degree two or less.

Let $p(x) = ax^2 + bx + c$ and $q(x) = dx^2 + ex + f$. Then

$$f'(x) = \left(\frac{p(x)}{q(x)}\right)' = \frac{p'(x)q(x) - p(x)q'(x)}{q^2(x)} = \frac{(2ax+b)(dx^2+ex+f) - (ax^2+bx+c)(2dx+e)}{q^2(x)}$$

$$= \frac{(ae-bd)x^2 + (2af-2cd)x + bf - ce}{q^2(x)}$$

Since the numerator of the derivative is a quadratic polynomial, there are at most two values of x at which $f'(x) = 0$. If we consider where $f'(x)$ is undefined, we see that this will be where $q(x) = 0$; since these points are not in the domain of $f(x)$, they cannot be local extrema.

Thus, $f(x)$ can have at most two local extrema.

Now, write $f'(x)$ as

$$f'(x) = \frac{R(x)}{q^2(x)}$$

so that $R(x)$ is a quadratic polynomial as shown above. Then

$$f''(x) = \frac{R'(x)q^2(x) - 2R(x)q(x)q'(x)}{q^4(x)} = \frac{R'(x)q(x) - 2R(x)q'(x)}{q^3(x)}.$$

Since $q(x)$ is quadratic, $q'(x)$ is linear. Similarly, $R(x)$ is quadratic, so $R'(x)$ is linear. Hence, $R'(x)q(x)$ is cubic, as is $R(x)q'(x)$. Hence the numerator of $f''(x)$ is cubic, and so $f''(x)$ has at most three roots. The derivative $f''(x)$ is undefined in the same places as $f(x)$, so there can be no inflection points at those points.

Thus, $f(x)$ can have at most three inflection points.

Four examples can be used to show that all values between 0 and these upper bounds can be attained.

The function

$$f(x) = \frac{1}{x}$$

has no inflection points or local extrema: the first derivative

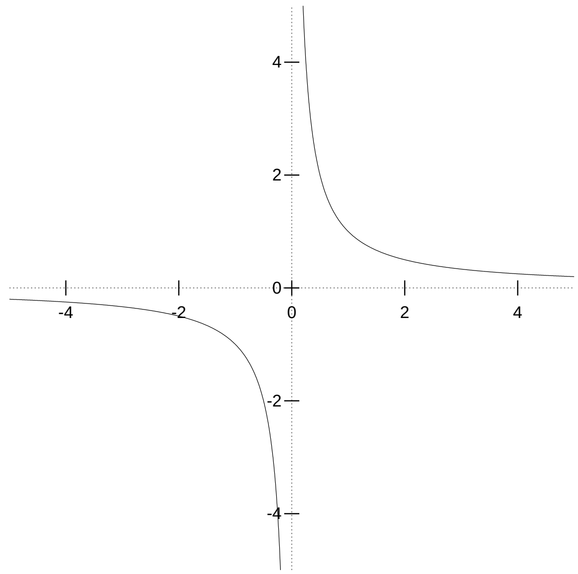
$$f'(x) = -\frac{1}{x^2}$$

and the second derivative

$$f''(x) = \frac{2}{x^3}$$

both are never equal to zero, and are undefined at exactly those values of x at which $f(x)$ is undefined.

Here is a graph of the function:



The function

$$f(x) = \frac{x}{x^2 - 1}$$

has first derivative

$$f'(x) = -\frac{x^2 + 1}{(x^2 - 1)^2}$$

and second derivative

$$f''(x) = \frac{2x(3 + x^2)}{(x^2 - 1)^3}.$$

We can see from the numerator of the first derivative that there are no solutions to the equation

$$f'(x) = 0$$

and hence, since local extrema of this function would give solutions to this equation, there must be no local extrema.

Looking at the second derivative, we can see that the only solution to the equation

$$f''(x) = 0$$

is $x = 0$. Keeping in mind that the function is undefined at 1 and -1, we check either side of zero (but in the interval $-1 \leq x \leq 1$) to see if there is a change of concavity at $x = 0$. We find that

$$f''\left(-\frac{1}{2}\right) = \frac{208}{27} > 0$$

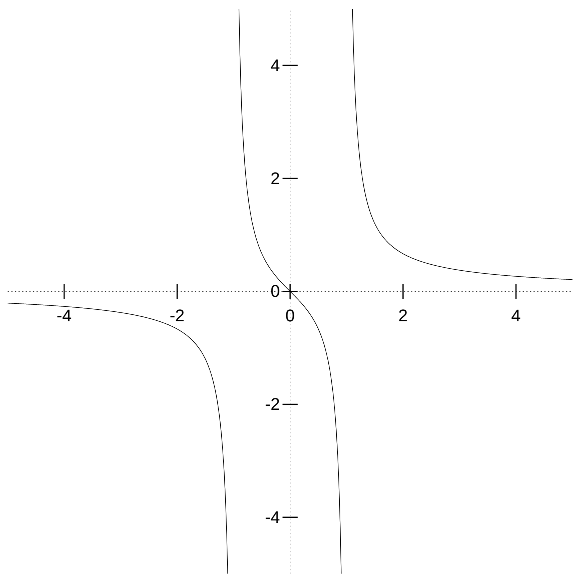
and that

$$f''\left(\frac{1}{2}\right) = -\frac{208}{27} < 0$$

Thus, we see that there is a change of concavity at $x = 0$, and so $x = 0$ is a point of inflection.

Hence, this function has no local extrema, and one point of inflection.

Here is a graph of this function, in which you can see the point of inflection at $x = 0$.



The function

$$f(x) = \frac{1}{x^2 + 1}$$

has first derivative

$$f'(x) = -\frac{2x}{(x^2 + 1)^2}$$

and second derivative

$$f''(x) = \frac{6x^2 - 2}{(1 + x^2)^3}.$$

We can see from the first derivative that the only solution to

$$f'(x) = 0$$

is $x = 0$. Noting that

$$f'(1) = -\frac{1}{2}$$

and

$$f'(-1) = \frac{1}{2},$$

we conclude that $f(x)$ changes from increasing to decreasing at $x = 0$, and hence there is a local maximum at $x = 0$. Thus, this function has exactly one local extrema.

The equation

$$f''(x) = 0$$

has solutions

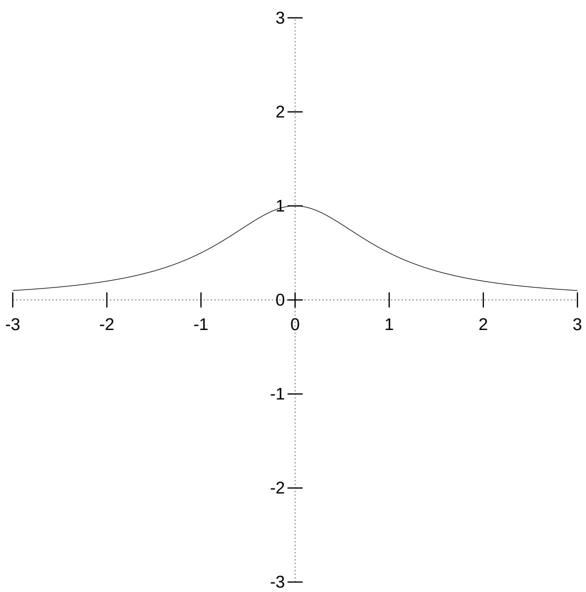
$$x = \pm\frac{1}{\sqrt{3}} \sim \pm 0.57735027.$$

By evaluating the second derivative in a few places, say

$$f''(-1) = \frac{1}{2}, f''(0) = -2, \text{ and } f''(1) = \frac{1}{2}$$

we can conclude that there is a change from concave up to concave down at $x = -\frac{1}{\sqrt{3}}$ and a change from concave down to concave up at $x = \frac{1}{\sqrt{3}}$. Thus, there are points of inflection at $x = -\frac{1}{\sqrt{3}}$ and $x = \frac{1}{\sqrt{3}}$.

Hence, this function has one local extrema, and two points of inflection. Here is a graph of the function showing these features.



Finally, let

$$f(x) = \frac{x}{x^2+1}$$

Then we have

$$f'(x) = \frac{1-x^2}{(1+x^2)^2}$$

and

$$f''(x) = \frac{2x(x^2-3)}{(1+x^2)^3}$$

We can see that there are two solutions, $x = 1$ and $x = -1$, to the equation

$$f'(x) = 0.$$

Noting that

$$f'(-2) = -\frac{3}{25} < 0, f'(0) = 1 > 0, \text{ and } f'(2) = -\frac{3}{25},$$

we see that the function changes from decreasing to increasing at $x = -1$ and from increasing to decreasing at $x = 1$. Thus, we have local extrema at $x = -1$ and $x = 1$.

We can see that there are three solutions, $x = 0$ and $x = \pm\sqrt{3}$, to the equation

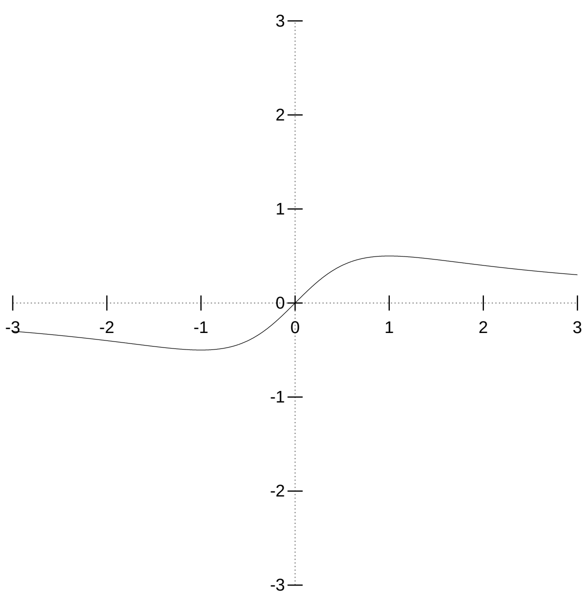
$$f''(x) = 0.$$

Noting that

$$f''(-2) = -\frac{4}{125}, f''(-1) = \frac{1}{2}, f''(1) = -\frac{1}{2}, \text{ and } f''(2) = \frac{4}{125},$$

we can see that there is a change of concavity at $x = -\sqrt{3}$, $x = 0$, and $x = sqrt3$, and hence there are points of inflection at these three points.

Hence this function has two local extrema, and three points of inflection. Here is a graph of the function, showing these features. Here is a graph of the function:



In conclusion, we have seen that a rational function with numerator and denominator of degree not more than two can have 0, 1, or 2 local extrema, and 0, 1, 2, or 3 points of inflection.