Writing Problem #2 Solution

The problem: For what values of a and b is

$$\lim_{x\to 0} \frac{\sqrt{ax+b}-3}{x} = 1\ ?$$

The solution: We may rewrite the limit as follows:

$$\lim_{x\to 0} \frac{\sqrt{ax+b}-3}{x} = \lim_{x\to 0} \frac{\sqrt{ax+b}-3}{x}\frac{\sqrt{ax+b}+3}{\sqrt{ax+b}+3} = \lim_{x\to 0} \frac{ax+b-9}{x(\sqrt{ax+b}+3)} = 1$$

If the limit of the numerator, i.e.

$$\lim_{x\to 0}(ax+b-9)$$

is not zero, then the quotient

$$\frac{ax+b-9}{x(\sqrt{ax+b}+3)}$$

will be unbounded (and thus not approach 1) as x approaches 0, since

$$\lim_{x\to 0} x(\sqrt{ax+b}+3) = 0.$$

Hence, it must be the case that

$$\lim_{x\to 0}(ax+b-9) = b-9 = 0$$

and so we conclude that $b = 9$.

Hence,

$$1 = \lim_{x\to 0} \frac{\sqrt{ax+9}-3}{x} = \lim_{x\to 0} \frac{ax+9-9}{x(\sqrt{ax+9}+3)} = \lim_{x\to 0} \frac{a}{\sqrt{ax+9}+3} = \frac{a}{\sqrt{9}+3} = \frac{a}{6}$$

and so $a = 6$.

Thus, $a = 6$ and $b = 9$.