Course Review for Math 124 - Winter 2007

- 1.2, 1.3, 1.5, 1.6, Appendix D, 10.1 Review of Precalculus Topics

  These sections were covered in the first week of the course, and were just a review of prerequisite material.

- 2.1 The Tangent and Velocity Problems

  This short section introduces two motivations for the development of the derivative: tangent lines and rates of change.

- 2.2 The Limit of a Function

  Here, the **limit** is introduced.

  You should be able to evaluate limits from graphs of functions as in problems 4-10.

- 2.3 Calculating Limits Using the Limit Laws

  In this section, a number of laws (i.e., rules) are introduced that allow us to evaluate quite a few limits.

  Problems 11-30 are especially good practice using these rules.

- 2.5 Continuity

  Continuity is one concept that makes working with limits simpler: if we know $f(x)$ is continuous for all x, then

  $$\lim_{x \to a} f(x) = f(a)$$

  for all a.

  Pay close attention to what continuity/discontinuity looks like in a graph (problems 3, 4). Problems 15-20 are good practice working with continuity of multi-part functions.

- 2.6 Limits at Infinity: Horizontal Asymptotes

  In this section, limits of the form

  $$\lim_{x \to \infty} f(x)$$

  are considered. If

  $$\lim_{x \to \infty} f(x) = L$$

  or

  $$\lim_{x \to -\infty} f(x) = L$$

  then we say that $y = L$ is a horizontal asymptote of $f(x)$.

  Lots of good problems for practice: 11-34.

- 2.7 Tangents, Velocities, and Other Rates of Change

  This minor section connects tangents, velocities and other rates of change to the limit idea.

- 2.8 Derivatives

  Finally, we get to the workhorse idea of the course: the **derivative**. In this section, the derivative is introduced as a limit:

  $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

  Be sure to spend some time practicing the evaluations of the derivative using the definition: e.g., problems 13-18.

- 2.9 The Derivative as a Function

  Here, we switch from thinking of the derivative as a single value (the slope of the tangent line) and give the name derivative to the function $f'(x)$ that gives us the slope of the tangent line at any point x.

  This section also introduces some of the graphical connections between $f(x)$ and $f'(x)$. See problems 5-12 for practice with this concept.

- 3.1 Derivatives of Polynomials and Exponential Functions

  In this section, the first derivative rules are developed, particularly the **power rule** and the fact that

  $$\frac{d}{dx} e^x = e^x.$$

  I particularly like problems 45, 54, and 59 from this section.

- 3.2 The Product and Quotient Rules

  Two mainstays of differentiation, the product and quotient rules are introduced here.

  I like problems 31, 33, and 41.

- 3.4 Derivatives of Trigonometric Functions

  The key facts from this section are

  $$\frac{d}{dx} \sin x = \cos x$$

  and

  $$\frac{d}{dx} \cos x = -\sin x.$$

  You should know how to find the derivatives of all the other trig functions ($\tan x$, $\sec x$, etc.) using these facts.

  Problem 33's not bad.

- 3.5 The Chain Rule

  The chain rule is the *pièce de résistance* of differentiation, allowing us to extend all the other rules to functions of unlimited complexity.

  The chain rule comes in two forms. The first, and the most commonly used so far in the course, is, in abbreviated form,

  $$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x).$$

  The second form, while equivalent, looks a bit different. Again in abbreviated form, it states that

  $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

  This can be interpreted as stating that derivatives can be treated, to a certain extent, like fractions.

  One of the most common applications of the chain rule is in the form of the **generalized power rule**:

  $$\frac{d}{dx}(f(x))^n = n(f(x))^{n-1}f'(x).$$

  The problems from this section are good practice for applying the rule.

- 3.6 Implicit Differentiation

  With all the derivative rules behind us, we look in this section toward extending our ability to apply them to other situations. In particular, we want to be able to discuss $\frac{dy}{dx}$ when y and x are related by an equation, but not necessarily by a function.

  Here, the key is to remember your differentiation rules, and in particular remember the chain rule, so that the derivative (with respect to x) of y^5 is

  $$5y^4\frac{dy}{dx}$$

  and not just

  $$5y^4.$$

  Problems 29, 35, 63 and 69 are good ones.

- 3.7 Higher Derivatives

  A short chapter, whose significance will appear in later courses (and to a limited extent later in this course).

  Problems 25 and 31 are the key ones for the mechanical aspects.

- 3.8 Derivatives of Logarithmic Functions

  The key fact in this section is that

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

  From this, we can differentiate any logarithm function, since

$$\log_b x = \frac{\ln x}{\ln b}.$$

  Another idea in this section is that of **logarithmic differentiation**. This means using the natural logarithm, and the logarithm rules, to simplify the work needed to differentiate certain expressions (for instance, we can avoid some applications of the product and quotient rules with this method). See problems 35-46.

- 3.10 Related Rates

  The key idea here is that **if two quantities are related by an equation, we can differentiate the equation with respect to time (t) to get an equation relating their rates of change**.

  The key challenge in these problems can be the process of converting the statement of the problem into an equation relating variables whose rates of change we want to know. So, use the recipe provided in lecture to help you.

  Problems that were not assigned but are good for more practice include 8, 17, 18, 37 and any others that sound interesting to you.

- 3.11 Linear Approximation

  A big idea generally, but only a little one in this course. It boils down to the fact that if f has a derivative at a point $x = a$, then we can approximate f near a with the tangent line to $y = f(x)$ at $x = a$:

$$f(x) \approx f(a) + f'(a)(x - a)$$

  for x near a.

  Problems 5-8 and 31-36 are good practice.

- 4.1 Maximum and Minimum Values

  This section introduces one of the most important applications of the derivative: to the finding of local maximum and minimum values of a function.

  For a function f on a closed interval I, we have a nice method for finding the **absolute maximum** and **absolute minimum** values:

  1. Find the value of f at the **critical values** of f in the interval I (i.e., where $f'(x) = 0$ or $f'(x)$ is undefined)

2. Find the value of f at the endpoints of I.
3. The largest of the values from steps 1 and 2 is the absolute maximum of f on I; the smallest value is the absolute minimum.

Lots of good practice problems: 31-62 are all good.

Problem 63 is good for working with parameters.

- 4.3 How Derivatives Affect the Shape of a Graph

  Here we look at the ideas of **increasing**, **decreasing**, **concave up**, **concave down** and inflection points and how they relate to the first and second derivatives.

  Problems 33-52 are all good practice.

- 4.4 Indeterminate Forms and L'Hospital's Rule

  This section gives us **l'Hospital's Rule** which allows us to say that

  $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

  under certain conditions: in particular, if $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ or $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$.

  This rule is very powerful, and a big help in evaluating limits. However, be careful with its use: it can only be applied if the numerator and denominator are both heading to zero, or both heading toward infinity (or minus infinity).

  Keep in mind as well that you may need to apply the rule repeatedly before you can make a conclusion, e.g.:

  $$\lim_{x \to \infty} \frac{e^x}{x^3}.$$

  Be sure that you have a $0/0$ or ∞/∞ form before apply l'Hospital's rule.

  Tons of practice problems: 5-62.

- 4.5 Summary of Curve Sketching

  This section puts the last few together; with all these tools, we can get quite detailed information about functions which can be used to sketch their graphs.

  Lots of good problems here, but they can each take quite a bit of work, so I'll just suggest a few: 5, 9, 11, 30, 33, 45, 49, 51.

- 4.7 Optimization Problems

  An extremely powerful application of the previous ideas is to **optimization**: situations where we want to make a quantity as large or as small as possible, usually with some constraints.

A lot of these problems involve some geometry, including the Pythagorean theorem, similar triangles, or volumes and surface areas of three-dimensional objects (cones, cylinders, etc.).

Like curve sketching, just one of these problems can take a lot of work. I'll just recommend the problems I put on the homework schedule: 5, 9, 13, 23, 35, 56.

- Parametric Calculus problems

  Throughout the course, there have been parametric problems in the homework. Here's a summary of the main ideas related to them.

  1. Rates of change

     If an object's position at time t is given by the parametric equations

     $$x = x(t), y = y(t)$$

     then the horizontal velocity of the object at time t is $x'(t)$ while the vertical velocity of the object at time t is $y'(t)$. The speed of the object at time t is then

     $$v(t) = \sqrt{(x'(t))^2 + (y'(t))^2}$$

  2. Tangent lines

     For a curve defined by the parametric equations $(x(t), y(t))$, the slope of the tangent line at $(x(a), y(a))$ is

     $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(a)}{x'(a)}.$$