The Problem: Suppose a projectile is launched from (0,0) with velocity $v_0$ at an angle $\theta$ above the positive $x$-axis. We want the projectile to make it over a wall that is $r$ units away horizontally, and $h$ units high. If gravity is the only force acting on the projectile, what is the minimum possible value of $v_0$ that will get the projectile over the wall?

Solution: Under the influence of gravity only, which results in vertical acceleration of $-g$, we have parametric equations for the projectile’s motion:

$$x = v_0 t \cos \theta, \quad y = v_0 t \sin \theta - \frac{1}{2} g t^2.$$

By solving for $x$, we can get the cartesian equation for the projectiles location

$$y = x \tan \theta - \frac{gx^2}{2v_0^2 \cos^2 \theta}.$$

If the velocity $v_0$ is the minimum which gets the projectile over the wall, the path of the projectile must pass through the point $(r, h)$. Hence, we know

$$h = r \tan \theta - \frac{gr^2}{2v_0^2 \cos^2 \theta}.$$

From this, we can solve for $v_0^2$:

$$v_0^2 = \frac{\frac{1}{2} gr^2}{\cos^2 \theta (r \tan \theta - h)}$$

Minimizing $v_0$ is equivalent to minimizing $v_0^2$, which is equivalent to maximizing the denominator above. Let

$$w = \cos^2 \theta (r \tan \theta - h)$$

We want to maximize $w$. Suppose

$$\frac{dw}{d\theta} = -2 \sin \theta \cos \theta (r \tan \theta - h) + r = 0.$$

Let $z = \sin \theta$. Then we can write

$$-2x \sqrt{1 - z^2} \left( r \frac{z}{\sqrt{1 - z^2}} - h \right) + r = 0$$

which simplifies to

$$4h^2 z^2 (1 - z^2) = 4r^2 z^4 - 4r^2 z^2 + r^2.$$

Let $\alpha = z^2$. Then we have

$$0 = 4(r^2 + h^2) \alpha^2 - 4(r^2 + h^2) \alpha + r^2.$$
The quadratic formula, and some simplification yields

\[ \alpha = \frac{1}{2} \pm \frac{h}{2\sqrt{r^2 + h^2}} \]

By some argument to be added later, we can conclude that

\[ \alpha = \frac{1}{2} + \frac{h}{2\sqrt{r^2 + h^2}} \]

Hence the optimal \( \theta \) has

\[ \sin \theta = \sqrt{\frac{1}{2} + \frac{h}{2\sqrt{r^2 + h^2}}} \]

and

\[ \cos \theta = \sqrt{\frac{1}{2} - \frac{h}{2\sqrt{r^2 + h^2}}} \]

and

\[ \tan^2 \theta = \frac{\frac{1}{2} + \frac{h}{2\sqrt{r^2 + h^2}}}{\frac{1}{2} - \frac{h}{2\sqrt{r^2 + h^2}}} = \frac{\sqrt{r^2 + h^2} + h}{\sqrt{r^2 + h^2} - h} \]

This then yields

\[ \tan \theta = \frac{\sqrt{r^2 + h^2} + h}{r} \]

Plugging this all into

\[ v_0^2 = \frac{\frac{1}{2}gr^2}{\cos^2 \theta (r \tan \theta - h)} \]

and simplifying, we get

\[ v_0^2 = \frac{gr^2}{\sqrt{r^2 + h^2} - h} = g(\sqrt{r^2 + h^2} + h) \]

so the optimal velocity \( v_0 \) is

\[ v_0 = \sqrt{g \left( \sqrt{r^2 + h^2} + h \right)} \]