Hints on the Donut Problem

Here are some steps toward a solution to this problem. You should work out the details.

1. Using the method of cylindrical shells, you can express the volume of the whole donut as

$$V = \int_{R-r}^{R+r} 4\pi x\sqrt{r^2 - (x-R)^2}\, dx$$

2. Do a substitution: let $u = x - R$ and rewrite V:

$$V = 4\pi \int_{-r}^{r} (u+R)\sqrt{r^2 - u^2}\, du$$

3. Split this integral into two:

$$V = 4\pi \int_{-r}^{r} \sqrt{r^2 - u^2}\, du + 4\pi \int_{-r}^{r} u\sqrt{r^2 - u^2}\, du$$

4. Evaluate these two integrals. The result is

$$V = 2\pi^2 Rr^2.$$

5. Do the same thing for the remaining half donut. The volume is

$$H = 4\pi R \int_{-r}^{0} \sqrt{r^2 - u^2}\, du + 4\pi \int_{-r}^{0} u\sqrt{r^2 - u^2}\, du$$

6. Again, evaluate these integrals. The result will be

$$H = \pi^2 Rr^2 - \frac{4}{3}\pi r^3.$$

7. Hence, the proportion that is remaining is

$$\frac{H}{V} = \frac{\pi^2 Rr^2 - \frac{4}{3}\pi r^3}{2\pi^2 Rr^2} = \frac{3\pi R - 4r}{6\pi R} = \frac{1}{2} - \frac{2r}{3\pi R}.$$