Math 125 C - Winter 2005
Mid-Term Exam Number One Solutions
January 27, 2005

1. (a) *Is* $\frac{e^x}{x}$ *an antiderivative of* $\frac{e^x}{x}\left(1-\frac{1}{x}\right)$ *? Explain.*

   Since
   $$\frac{d}{dx}\frac{e^x}{x}=\frac{e^x}{x}\left(1-\frac{1}{x}\right)$$
   the answer is yes.

   (b) *Suppose* $f(t)=2g(t)+7$ *and*
   $$\int_{-3}^{5} g(t)\,dt=9.$$
   Find
   $$\int_{-3}^{5} f(x)\,dx.$$
   $$\int_{-3}^{5} f(x)\,dx=\int_{-3}^{5} f(t)\,dt=\int_{-3}^{5}(2g(t)+7)\,dt$$
   $$=2\int_{-3}^{5} g(t)\,dt+\int_{-3}^{5} 7\,dt=2\cdot 9+7(5-(-3))=74.$$

2. *Evaluate the following integrals:*

   (a) $$\int_0^1\left(x^5-2x^4+\frac{1}{2}x^3-2x\right)\,dx$$
   $$=\left(\frac{1}{6}x^6-\frac{2}{5}x^5+\frac{1}{8}x^4-x^2\right)\Big|_0^1=-\frac{133}{120}.$$

   (b) $$\int_{-6}^{6}|7+4t|\,dt$$
   The easiest way to evaluate this integral is to interpret it as an area. It is the sum of the areas of two triangles: one has width $-\frac{7}{4}-(-6)$ and height 17 while the other has width $6-(-\frac{7}{4})$ and height 31. Hence the integral is equal to
   $$\frac{1}{2}(17)\left(-\frac{7}{4}+6\right)+\frac{1}{2}(31)\left(6+\frac{7}{4}\right)=156.25.$$

   (c) $$\int_e^{e^2}\frac{1}{x(\ln x)^5}\,dx$$
   Let $u=\ln x$ so the $du=\frac{1}{x}\,dx$ and we have
   $$\int_e^{e^2}\frac{1}{x(\ln x)^5}\,dx=\int_1^2\frac{1}{u^5}\,du=-\frac{1}{4}u^{-4}\Big|_1^2=\frac{15}{64}.$$

3. *Evaluate the following integrals:*

(a) $$\int \frac{1+\cos x}{\sin x + x}\, dx$$

Let $u = \sin x + x$ so that $du = (\cos x + 1)\, dx$ and the integral equals

$$\int \frac{du}{u} = \ln|u| + C = \ln|\sin x + x| + C.$$

(b) $$\int e^x \sin\left(e^x + 8\right)\, dx$$

Let $u = e^x + 8$ so that $du = e^x\, dx$. Then the integral equals

$$\int \sin u\, du = -\cos u + C = -\cos(e^x + 8) + C.$$

(c) $$\int \left(x^3 - x\right)\sqrt{x^2-1}\, dx$$

There are a number of ways to handle this integral. One is to expand the integrand to get

$$\int \left(x^3 - x\right)\sqrt{x^2-1}\, dx = \int x^3\sqrt{x^2-1}\, dx - \int x\sqrt{x^2-1}\, dx = A - B, \text{ say.}$$

Using the same substitution for both integrals, let $u = x^2 - 1$ so $x^2 = u + 1$ and $du = 2x\, dx$, or $\frac{1}{2}du = x\, dx$. Then

$$A = \frac{1}{2}\int (u+1)\sqrt{u}\, du = \frac{1}{2}\int (u^{3/2} + u^{1/2})\, du = \frac{1}{5}u^{5/2} + \frac{1}{3}u^{3/2} + C$$

$$= \frac{1}{5}(x^2-1)^{5/2} + \frac{1}{3}(x^2-1)^{3/2} + C.$$

and

$$B = \frac{1}{2}\int u^{1/2}\, du = \frac{1}{3}u^{3/2} + C = \frac{1}{3}(x^2-1)^{3/2} + C.$$

So, the integral equals

$$A - B = \frac{1}{5}(x^2-1)^{5/2}.$$

4. *Consider the region bounded by $y = 8 - x^2$ and $y = x^2 - 2x - 4$.*

(a) *Find the area of the region.*

Determining the intersection of the two curves we find

$$x^2 - 2x - 4 = 8 - x^2$$

$$x^2 - x - 6 = (x-3)(x+2) = 0$$

so the intersections are at $x = -2$ and $x = 3$. Evaluating the two functions at $x = 0$, we find that $8 - x^2$ is the "upper function", so the area is

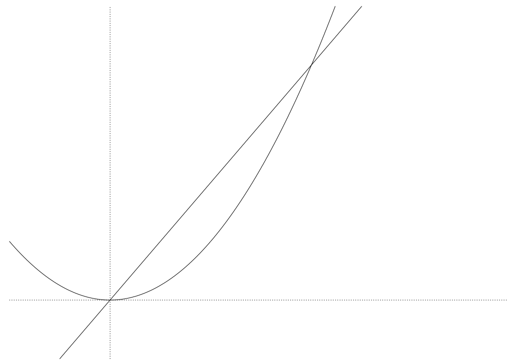
$$\int_{-2}^{3} (8 - x^2 - (x^2 - 2x - 4))\, dx = \int_{-2}^{3} (12 + 2x - 2x^2)\, dx = \left(12x + x^2 - \frac{2}{3}x^3\right)\Big|_{-2}^{3} = \frac{125}{3}.$$

(b) *Set up but DO NOT EVALUATE an integral representing the volume of the solid of revolution formed by revolving this region about the line $x = 5$.*

Using the cylindrical shells method, the volume is

$$\int_{-2}^{3} 2\pi(5-x)(8-x^2-(x^2-2x-4))\,dx = \int_{-2}^{3} 2\pi(5-x)(12+2x-2x^2)\,dx.$$

5. *A region in the first quadrant is bounded by two curves: $y = x^3$, and a line with a positive slope passing through the origin.*



(a) *If the area of this region is 5, what is the slope of the line?*

Let m be the slope of the line. Then the line is $y = mx$ and it intersects the curve when

$$mx = x^3$$

$$x = \sqrt{m} \text{ or } x = 0.$$

Hence the area of the region is given by the integral

$$\int_0^{\sqrt{m}} (mx - x^3)\,dx = \left(\frac{1}{2}mx^2 - \frac{1}{4}x^4\right)\Big|_0^{\sqrt{m}} = \frac{1}{4}m^2.$$

So,

$$\frac{1}{4}m^2 = 5$$

and so

$$m = \sqrt{20} = 4.4721....$$

(b) *If the volume of the solid of revolution created by revolving this region about the x-axis is 2, what is the slope of the line?*

As in (a), the line is $y = mx$ and it intersects $y = x^3$ at $x = 0$ and $x = \sqrt{m}$. So, using the "washer" method, the volume is given by

$$\int_0^{\sqrt{m}} \pi((mx)^2 - (x^3)^2)\,dx = \pi \int_0^{\sqrt{m}} \left(\frac{1}{3}m^2x^3 - \frac{1}{6}x^6\right)\Big|_0^{m} = \frac{4}{21}\pi m^{\frac{7}{2}} = 2.$$

Solving this last equation for m, we have

$$m = \left(\frac{21}{2\pi}\right)^{\frac{2}{7}} = 1.411645....$$