Math 125 C - Winter 2005
Mid-Term Exam Number One Solutions
January 27, 2005

1. (a) *Is* $\frac{e^x}{x}$ *an antiderivative of* $\frac{e^x}{x}\left(\frac{1}{x^2}-1\right)$ *? Explain.*

   Since
   $$\frac{d}{dx}\frac{e^x}{x} = \frac{e^x}{x}\left(1-\frac{1}{x}\right) \neq \frac{e^x}{x}\left(\frac{1}{x^2}-1\right)$$
   the answer is no.

   (b) *Suppose* $f(t) = 3g(t) - 6$ *and*
   $$\int_{-3}^{5} g(t)\,dt = 12.$$
   Find
   $$\int_{-3}^{5} f(x)\,dx.$$
   $$\int_{-3}^{5} f(x)\,dx = \int_{-3}^{5} f(t)\,dt = \int_{-3}^{5}(3g(t)-6)\,dt$$
   $$= 3\int_{-3}^{5} g(t)\,dt - \int_{-3}^{5} 6\,dt = 3\cdot 12 - 6(5-(-3)) = -12.$$

2. *Evaluate the following integrals:*

   (a) $$\int_0^1 \left(x^5 - 3x^4 + \frac{1}{2}x^3 - x\right)dx$$
   $$= \left(\frac{1}{6}x^6 - \frac{3}{5}x^5 + \frac{1}{8}x^4 - \frac{1}{2}x^2\right)\Big|_0^1 = -\frac{97}{120} = -0.8083333333....$$

   (b) $$\int_{-5}^{5} |3+5t|\,dt$$
   The easiest way to evaluate this integral is to interpret it as an area. It is the sum of the areas of two triangles: one has width $-\frac{3}{5}-(-5)$ and height 22, the other has width $5-(-\frac{3}{5})$ and height 28. Hence the integral is equal to
   $$\frac{1}{2}(22)\left(-\frac{3}{5}+5\right)+\frac{1}{2}(28)(5+\frac{3}{5}) = \frac{634}{5} = 126.8.$$

   (c) $$\int_e^{e^2} \frac{1}{x(\ln x)^4}\,dx$$
   Let $u = \ln x$ so the $du = \frac{1}{x}\,dx$ and we have
   $$\int_e^{e^2} \frac{1}{x(\ln x)^4}\,dx = \int_1^2 \frac{1}{u^4}\,du = -\frac{1}{3}u^{-3}\Big|_1^2 = -\frac{7}{24}.$$

3. *Evaluate the following integrals:*

(a) $$\int \frac{1+\cos x}{\sin x + x}\, dx$$

Let $u = \sin x + x$ so that $du = (\cos x + 1)\, dx$ and the integral equals

$$\int \frac{du}{u} = \ln|u| + C = \ln|\sin x + x| + C.$$

(b) $$\int e^x \sin\left(e^x - 3\right)\, dx$$

Let $u = e^x - 3$ so that $du = e^x\, dx$. Then the integral equals

$$\int \sin u\, du = -\cos u + C = -\cos(e^x - 3) + C.$$

(c) $$\int \left(x^3 + x\right)\sqrt{x^2+1}\, dx$$

There are a number of ways to handle this integral. One is to expand the integrand to get

$$\int \left(x^3 + x\right)\sqrt{x^2+1}\, dx = \int x^3\sqrt{x^2+1}\, dx + \int x\sqrt{x^2+1}\, dx = A + B, \text{ say.}$$

Using the same substitution for both integrals, let $u = x^2 + 1$ so $x^2 = u - 1$ and $du = 2x\, dx$, or $\frac{1}{2}du = x\, dx$. Then

$$A = \frac{1}{2}\int (u-1)\sqrt{u}\, du = \frac{1}{2}\int (u^{3/2} - u^{1/2})\, du = \frac{1}{5}u^{5/2} - \frac{1}{3}u^{3/2} + C$$

$$= \frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + C.$$

and

$$B = \frac{1}{2}\int u^{1/2}\, du = \frac{1}{3}u^{3/2} + C = \frac{1}{3}(x^2+1)^{3/2} + C.$$

So, the integral equals

$$A + B = \frac{1}{5}(x^2+1)^{5/2}.$$

4. *Consider the region bounded by $y = 10 - x^2$ and $y = x^2 - 2x + 6$.*

(a) *Find the area of the region.*

Determining the intersection of the two curves we find

$$x^2 - 2x + 6 = 10 - x^2$$

$$x^2 - x - 2 = (x-2)(x+1) = 0$$

so the intersections are at $x = -1$ and $x = 2$. Evaluating the two functions at $x = 0$, we find that $10 - x^2$ is the "upper function", so the area is

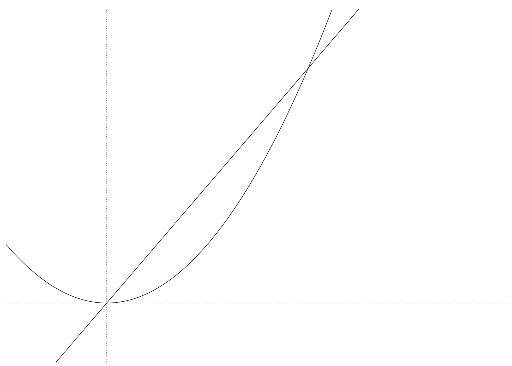
$$\int_{-1}^{2} (10 - x^2 - (x^2 - 2x + 6))\, dx = \int_{-1}^{2} (4 + 2x - 2x^2)\, dx = \left(4x + x^2 - \frac{2}{3}x^3\right)\Big|_{-1}^{2} = 9.$$

(b) *Set up but DO NOT EVALUATE an integral representing the volume of the solid of revolution formed by revolving this region about the line $x = 5$.*

Using the cylindrical shells method, the volume is

$$\int_{-1}^{2} 2\pi(5-x)(10-x^2-(x^2-2x+6))\,dx = \int_{-1}^{2} 2\pi(5-x)(4+2x-2x^2)\,dx.$$

5. *A region is bounded by two curves: $y = x^2$, and a line with a positive slope passing through the origin.*



(a) *If the area of this region is 1, what is the slope of the line?*

Let m be the slope of the line. Then the line is $y = mx$ and it intersects the curve when

$$mx = x^2$$

$$x = m \text{ or } x = 0.$$

Hence the area of the region is given by the integral

$$\int_0^m (mx - x^2)\,dx = (\frac{1}{2}mx^2 - \frac{1}{3}x^3)\Big|_0^m = \frac{1}{6}m^3.$$

So,

$$\frac{1}{6}m^3 = 1$$

and so

$$m = 6^{1/3} = 1.81712...$$

(b) *If the volume of the solid of revolution created by revolving this region about the x-axis is 1, what is the slope of the line?*

As in (a), the line is $y = mx$ and it intersects $y = x^2$ at $x = 0$ and $x = m$. So, using the "washer" method, the volume is given by

$$\int_0^m \pi((mx)^2 - (x^2)^2)\,dx = \pi \int_0^m \left(\frac{1}{3}m^2x^3 - \frac{1}{5}x^5\right)\Big|_0^m = \frac{2}{15}\pi m^5 = 1.$$

Solving this last equation for m, we have

$$m = \left(\frac{15}{2\pi}\right)^{\frac{1}{5}} = 1.190096....$$