Completing the Square

Completing the square is a technique for re-formatting certain algebraic expressions. The general idea is the following. Any time you have a quadratic expression in a variable, say x, such as $x^2 + Ax + B$, it is possible to express it in the form $(x + C)^2 + D$. The reasons for doing this are many, but we should remember that there is no inherent advantage of one form over another: they are merely means to an end.

To change $x^2 + Ax + B$ into the form $(x + C)^2 + D$ is to *c*omplete the square. The general method can be expressed by the equation

$$x^2 + Ax + B = \left(x + \frac{A}{2}\right)^2 - \left(\frac{A}{2}\right)^2 + B.$$

Exercise: Check this equality by expanding ("FOILing") the right-hand side.

If we now match up $\dfrac{A}{2}$ with C and $B - \left(\frac{A}{2}\right)^2$ with D, then we see we have it in the form we want:

$$x^2 + Ax + B = \left(x + \frac{A}{2}\right)^2 - \left(\frac{A}{2}\right)^2 + B = (x + C)^2 + D.$$

For example,

$$x^2 + 5x + 7 = \left(x + \frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 7 = \left(x + \frac{5}{2}\right)^2 + \frac{3}{4}.$$

Here is another example, showing how we can use completing the square to put the equation of a circle into a useful form. Starting with

$$x^2 - 4x + y^2 + 9y = 50$$

we complete the square on the x terms and the y terms to get:

$$(x - 2)^2 - 4 + \left(y + \frac{9}{2}\right)^2 - \frac{81}{4} = 50$$

which we can rewrite finally as

$$(x - 2)^2 + \left(y + \frac{9}{2}\right)^2 = \frac{297}{4}.$$

In this form, we can see that our equation is the equation of the circle with center $(2, -\frac{9}{2})$ and radius $\sqrt{\dfrac{297}{4}} = \dfrac{\sqrt{297}}{2}$.