Math 120A - Spring 2004
Mid-Term Exam Number One
April 22, 2004
Solutions

1. *A train runs along a straight track from the city of Springfield to the city of Alberta. Alberta is located 40 miles south and 35 miles east of Springfield.*

   (a) *The tiny nation of Circlvania is a 15 mile radius circle centered 17 miles south of Springfield. For what distance of the train's trip will the train be in Circlvania?*

   Solution: We first want to impose a coordinate system. There are several reasonable choices. One is to put the center of the circle (i.e., the center of Circlvania) at the origin $(0, 0)$. Then Springfield is at the point $(0, 17)$, and Alberta is at $(35, -23)$.

   The train's track is the line passing through $(0, 17)$ and $(35, -23)$. This line has slope

$$m = \frac{17 - (-23)}{0 - 35} = \frac{40}{-35} = -\frac{8}{7}$$

   and so it has equation

$$y = -\frac{8}{7}x + 17.$$

   The circle that is the border of Circlvania has equation

$$x^2 + y^2 = 15^2 = 225$$

   We are interested in the intersection of the line with the circle, so we solve the equation

$$x^2 + (-\frac{8}{7}x + 17)^2 = 225$$

   for x. Expanding and simplifying, we have

$$\frac{113}{49}x^2 - \frac{272}{7}x + 64 = 0$$

   so, using the quadratic formula, we have

$$x = \frac{\frac{272}{7} \pm \sqrt{\left(\frac{272}{7}\right)^2 - 4\left(\frac{113}{49}\right)(64)}}{2\frac{113}{49}}$$

   so we can conclude that our two solutions are

$$x = 1.85023050407... \text{ and } x = 14.999327018....$$

   These are the x-coordinates of the points where the train enters and exits (respectively) Circlvania. The y-values of these points are

$$y = 14.885450852... \text{ and } y = -0.1420880206...$$

   and the distance between the two points (i.e., the distance that the train travels in Circlvania) is

$$\sqrt{(1.85023050407 - 14.999327018)^2 + (14.885450852 - (-0.1420880206))^2}$$

$$= 19.968116178 \text{ miles.}$$

(b) *How close does the train get to the center of Circlvania?*

Solution: Using the same coordinate system as in part (a), we construct a line perpendicular to the train's track through the origin. Since the track is represented by the line

$$y = -\frac{8}{7}x + 17$$

the perpendicular line that passes through the origin is

$$y = \frac{7}{8}x.$$

The intersection of these lines is the nearest point on the track to the origin:

$$-\frac{8}{7} + 17 = \frac{7}{8}x$$

$$17 = \left(\frac{7}{8} + \frac{8}{7}\right)x = \frac{113}{56}x$$

$$x = \frac{952}{113}.$$

The y-coordinate of this point is

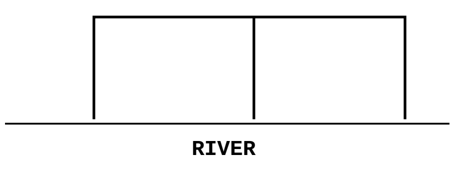
$$y = \frac{7}{8}\frac{952}{113} = \frac{833}{113}$$

and the distance from this point to the center of Circlvania (i.e., the origin) is

$$\sqrt{\left(\frac{952}{113}\right)^2 + \left(\frac{833}{113}\right)^2} = 11.194578333 \text{ miles.}$$

2. *A farmer wants to make a rectangular enclosure with 560 meters of fencing. One side of the rectangle is bounded by a river, so no fencing is needed on that side of the rectangle. Also, she wants to use some fencing to split the enclosure into two compartments as shown in the figure.*

What dimensions should she use for the enclosure to get the maximum possible area?



Solution: Let x be the length of a vertical side of the enclosure and y be the length of a horizontal side. Then the quantity we want to make as large as possible is the area

$$A = xy$$

Now, the amount of fencing needed to make the enclosure is

$$3x + y$$

so

$$560 = 3x + y$$

Solving for y, we have

$$y = 560 - 3x$$

so we can rewrite

$$A = xy = x(560 - 3x) = -3x^2 + 560x$$

Note that this is a quadratic function with a negative leading coefficient, so its maximum value will occur at its vertex. To find the vertex, we can put it into vertex form by completing the square:

$$A = -3x^2 + 560x = -3\left(x^2 - \frac{560}{3}x\right) = -3\left(\left(x - \frac{280}{3}\right)^2 - \left(\frac{280}{3}\right)^2\right) = -3\left(\left(x - \frac{280}{3}\right)^2\right) + 3\left(\frac{280}{3}\right)^2$$

From this, we can see that the largest area the enclosure can have is $3\left(\frac{280}{3}\right)^2 = \frac{78400}{3} = 26133.3333...$ square meters, and this will happen with dimensions

$$x = \frac{280}{3} = 93.33333... \text{ meters and } y = 280 \text{ meters.}$$

3. *Let* $f(x) = 4x^2 + 3$.

   (a) *Find the expression for*

   $$\frac{f(x+h) - f(x)}{h}$$

   and simplify as much as possible.
   Solution:

   $$\frac{f(x+h) - f(x)}{h} = \frac{4(x+h)^2 + 3 - (4x^2 + 3)}{h} = \frac{4(x^2 + 2xh + h^2) + 3 - 4x^2 - 3}{h} =$$

   $$\frac{4x^2 + 8xh + 4h^2 + 3 - 4x^2 - 3}{h} = \frac{8xh + 4h^2}{h} = 8x + 4h.$$

   (b) *Let* $g(x) = x + 1$. *Find all solutions to the equation* $g(f(x)) = f(g(x))$.
   Solution:

   $$g(f(x)) = f(x) + 1 = 4x^2 + 3 + 1 = 4x^2 + 4$$

   and

   $$f(g(x)) = f(x+1) = 4(x+1)^2 + 3 = 4(x^2 + 2x + 1) + 3 = 4x^2 + 8x + 4 + 3 = 4x^2 + 8x + 7$$

   so if

   $$f(g(x)) = g(f(x))$$

   then

   $$4x^2 + 4 = 4x^2 + 8x + 7$$

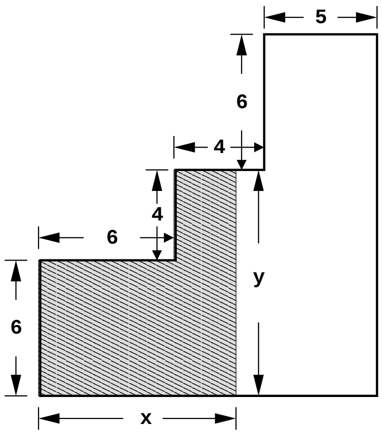
   which simplifies to

   $$-3 = 8x$$

   or

   $$x = -\frac{3}{8}$$

4. *Pizzeria Stairsteppo makes a pizza shaped as shown below. Alice wants only a portion of the pizza and does so by making a vertical cut through the pizza and taking the shaded portion. Let* x *be the bottom length of Alice's portion and* y *be the length of the cut as shown in the figure.*



(a) *Find a formula for y as a multipart function of x, for $0 \le x \le 15$.*
Solution:

$$y = \begin{cases} 6 & \text{if } 0 \le x \le 6, \\ 10 & \text{if } 6 < x \le 10, \\ 16 & \text{if } 10 < x \le 15 \end{cases}$$

The value of y at $x = 6$ could instead be 10, and the value of y at $x = 10$ could instead be 16, so there are several acceptable answers.

(b) *Find a formula for the area of Alice's portion as a multipart function of x, for $0 \le x \le 15$.*
Solution:

$$\text{area} = \begin{cases} 6x & \text{if } 0 \le x \le 6, \\ 36 + 10(x-6) & \text{if } 6 < x \le 10, \\ 76 + 16(x-10) & \text{if } 10 < x \le 15 \end{cases}$$

$$= \begin{cases} 6x & \text{if } 0 \le x \le 6, \\ 10x - 24 & \text{if } 6 < x \le 10, \\ 16x - 84 & \text{if } 10 < x \le 15 \end{cases}$$