5.5 Special Exponential Models

1. The bombardment of the upper atmosphere by cosmic rays converts nitrogen to a radioactive isotope of Carbon-14, with a half-life of 5730 years. Vegetation absorbs carbon dioxide and animals assimilate Carbon-14 through the food chain. When a plant or animal dies, it stops replacing its carbon and the amount of Carbon-14 begins to decrease through a radioactive decay model. By detecting the percentage of remaining Carbon-14 as compared to the normal percentage in the living plant or animal, one is able to date the age of the item.

   (a) A humanoid skull was found in a cave in South Africa along with the remains of a campfire. Archaeologists believe the age of the skull to be the same as the campfire. It is determined that 2% of the original amount of Carbon-14 remains in the burnt wood. Estimate the age of the skull.

   (b) An antique dealer in New York claims to have a piece of scratch paper used by Columbus during his voyage to America in 1492. The paper is made of cotton fibers. If the cotton which was used died 505 years ago, what percentage of the Carbon-14 would be left? After testing the paper, the fraction of Carbon-14 left is 97% of the amount originally present. Determine when the cotton used to make the paper died.

   (c) The half-life of the element Michaelonium is 4 hours. Find a constant a so that the function $f(x) = e^{ax}$ gives the fraction of Michaelonium present after x hours.

2. Iodine-131 is a radioactive material that decays according to the law $A(t) = A_o e^{-0.087t}$, where A_o is the initial amount present and $A(t)$ is the amount present at time t (in days).

   (a) What is the half-life of iodine-131 (i.e. when is the amount equal to $\frac{1}{2}A_o$)?

   (b) How long does it take 100 grams of iodine-131 to decay to 10 grams?

   (c) After the fallout at Chernobyl the hay in Austria was contaminated by iodine-131. If you feel it is alright to feed the hay to cows when no more than 10% of the iodine-131 remains, how long do the farmers need to wait to use this hay?

3. If you have 32 grams of Oxygen gas O_2 in a container at 27 ^oC, then these molecules move around at various speeds. If you specify a speed s, the function

$$p(s) = (3.66637 \times 10^{-6})s^2 e^{-(6.4146\times 10^{-6})s^2}$$

   gives the probability that a gas molecule in the container is moving at speed s.

   (a) Sketch the graph of $p(s)$ using a graphing device.

   (b) What is the most probable speed of a molecule in the container?

   (c) Find the speed(s) for which there is a 0.2% chance.

4.* The *TurboOrbit* golfball is being marketed as the worlds most bouncy ball. An independent test lab has modeled the bounce characteristics when dropped from a height of 64 feet using the function $b(t)$ which is the height of the ball above the ground (in feet) t seconds after release. The mathematical model is $b(t) = (1.2)^{-t} y(t)$, where $y(t)$ is the multipart function

$$y(t) = \begin{cases} -16t^2 + 64 & 0 \le t \le 2 \\ -16(t-4)^2 + 64 & 2 \le t \le 6 \\ -16(t-8)^2 + 64 & 6 \le t \le 10 \end{cases}$$

(a) Sketch the graph of the function $y(t)$ on the domain $0 \leq t \leq 10$.

(b) Sketch the graph of the function $y = (1.2)^{-t}$ on the domain $0 \leq t \leq 10$.

(c) Sketch the graph of $b(t)$ on the domain $0 \leq t \leq 10$.

(d) During the first 10 seconds, using a graphing device to approximate how much time the ball is less than 10 feet high.

5. Guests will begin arriving for a big party in 60 minutes and you suddenly realize you don't have enough ice. You quickly fill some ice cube trays with cold tap water and throw them in the freezer. Concerned about whether they'll be frozen in time, you take the following measurements: The temperature of the freezer is 22^oF, the cold tap water is 46^oF, and after 15 minutes the water in the trays has cooled to 42^oF.

   (a) *Newton's Law of Cooling* says

   $$T(t) = T_m + (T_o - T_m)e^{-kt},$$

   where $T(t)$ is the temperature of an object at time t; the object is in a medium which has constant temperature T_m; T_o is the initial temperature of the object (at time $t = 0$) and k is a constant which depends on the characteristics of the object. Assume the water in the trays cools according to this law. Will the water cool to 32^oF before the 60 minutes are up? If the H_2O in the trays needs to be below 32^oF for 12 minutes in order to freeze, when will the ice be ready?

   (b) Assume the temperature of the water in the trays will cool according to a linear model. What is the temperature of the H_2O in the trays when the party begins? If the H_2O in the trays needs to be below 32^oF for 12 minutes in order to freeze, when will the ice be ready?

   (c) Sketch a plot of the two temperature models in a. and b. in the same coordinate system (where the horizontal axis is time and the vertical axis is temperature in ^oF). Where do the two graphs intersect each other?

   (d) Using a graphing device, determine the time interval when the model $T(t)$ predicts a temperature lower than the linear model. During this time interval, what is the largest temperature difference between the linear model and the $T(t)$?

6.* The retina is a thin layer of cells which lines the back of the eye. Its job is to convert incoming light into electrical signals which the brain can interpret. Sometimes the retina tears and peels off. To fix this, the eye is emptied of fluid and filled with gas. The gas pushes the retina back in place. Reattachment of the retina takes at least three days. In that time, gases diffuse between the eye and the blood. If the amount of gas in the eye is less than A_{min}, the retina will not reattach, while if it is higher than A_{max}, the eye will suffer further damage (gross!). Ophthalmologists have asked you to figure out how much gas they should inject into the eye. [1]

[1] I am indebted to Rebecca Tyson for this problem.

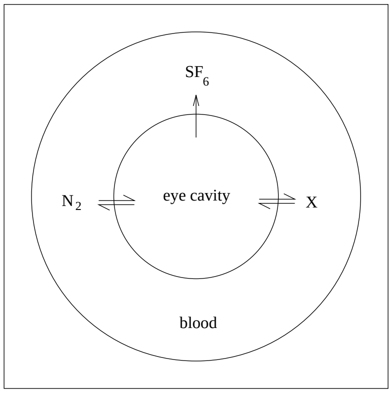


Figure 1: Compartmental Model of Eye.

	SF_6	N_2	X
amount of gas in eye	$y_1(t)$	$y_2(t)$	$y_3(t)$
amount of gas in blood	b_1	b_2	b_3
amount of gas in eye at $t = 0$	a_1	a_2	a_3
exponential decay rate	r_1	r_2	r_3

(a) You decide to model the situation with two compartments: eye and blood (see figure). You are told that the important gases are SF_6, N_2 and X. You know that the amount of each gas in the eye follows a decaying exponential relationship, which asymptotes at the partial pressure in the blood. Using the symbols listed in the table, write an equation for the amount of each gas as a function of time. Assume all constants are positive. (Hint: the general equation is $y(t) = A_o e^{rt} + b$, you need to find A_o, r and b in terms of the symbols in the table).

(b) The total amount of gas in the eye, $y(t)$, is the sum of the all the individual gas amounts. Write the equation for $y(t)$.

(c) On one graph, sketch $y_2(t)$ for the cases: (i) $a_2 < b_2$; (ii) $a_2 > b_2$; (iii) $a_2 = b_2$ and state which way the gas is diffusing.

(d) Experiments have shown that the amount of gas X, which is modelled by $y_3(t)$, has an asymptote at $y = b_3$. From these experiments we know $y_3(6\,\text{hrs}) = b_3 + 0.1$. Find an equation for r_3 in terms of a_3 and b_3.

(e) If $a_2 = b_2$, $a_3 = b_3$ and $a_1 > 0$, write the equation for $y(t)$. Solve this equation to find and expression for the time t_{min} at which $y = A_{min}$ and the time t_{max} at which $y = A_{max}$. What conditions must you impose on b_2 and b_3 so that t_{max} and t_{min} exist?

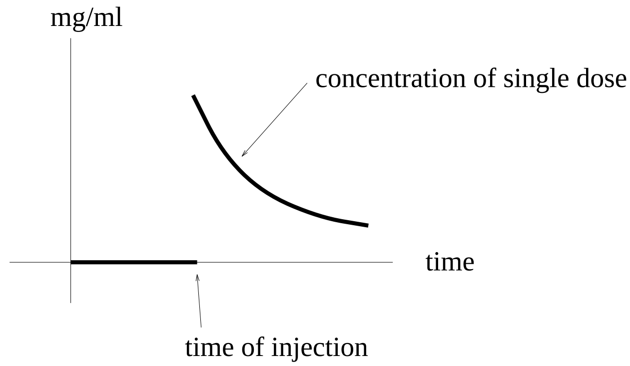
(f) Given

$$r_1 = 0.01\frac{1}{\text{hr}} \qquad b_1 = 0 \qquad a_1 = 233$$
$$r_2 = 0.02\frac{1}{\text{hr}} \qquad b_2 = 573 \qquad a_2 = 410$$
$$b_3 = 132 \qquad a_3 = 132$$
$$A_{min} = 775 \quad A_{max} = 795$$

write the equation for y(t). Use a graphing device to approximate t_{min} (initial guess: $t_1 = 50\text{hr}$). How would you check if $y(t)$ is ever greater than A_{max}?

(g) Should a_1 and a_2 be increased or decreased in order for $y(t)$ to better satisfy the ophthalmologists' criteria? (Hint: a clear statement of the criteria is that $y(t) < A_{max}$ for all t **and** $y(t) > A_{min}$ for $0 \leq t \leq 72\text{hr}$.)

7.* The concentration in the blood resulting from a single injected dose of the drug *Superthyamine* decreases with time as the drug is eliminated from the body. We will assume the drug is diffused so rapidly throughout the bloodstream that, for all practical purposes, it reaches its fullest concentration instantaneously. Thus, the concentration jumps from the initial concentration of zero at the time of injection to a higher value.



The model for the concentration of the single drug dose after injection decreases as a function of time according to the formula: Pe^{-kt}, for some constants P, k. Now, there are two problem a physician faces when ordering a regular treatment of this drug:

- As with most drugs, there is a concentration below which the drug will be ineffective and a concentration above which the drug becomes dangerous.
- When a patient starts taking multiple doses, each individual dose will decay according to the above model, but there will be a built up accumulation of the drug in the patients system.

Assume our patient takes equal sized doses of Superthyamine at times $t = 0, 4, 8$ hours. With the first injection the concentration in the patients blood is initially 2 mg/ml. After 2 hours the concentration is monitored to be 1.5 mg/ml. The minimum effective concentration of Superthyamine is 1.4 mg/ml and the maximum safe concentration is 3 mg/ml.

(a) Let $c(t)$ be the concentration of Superthyamine in the patients blood after t hours. Find a formula for $c(t)$ for the first 12 hours and sketch the graph of this function.

(b) During the first 12 hours, how much of the time is the drug treatment being ineffective and how of the time is it dangerous?

8. The Washington Legislature decides that Math Teaching Assistants deserve a big raise. The final budget contains a formula for the monthly salary of a TA:

$$S(t) = \$1050e^{-0.009t(t-21)},$$

where t represents months after September 1995. The formula is seriously flawed. Let's see why.

(a) What is a TA monthly salary Sept. 1995 and Sept. 1996?

(b) When is a TA monthly salary \$2,000?

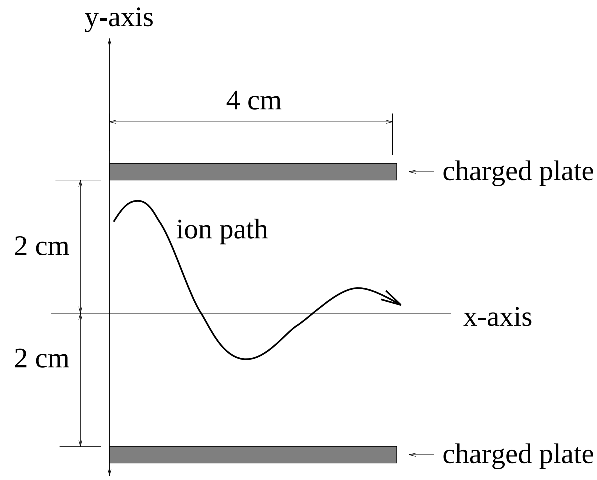
(c) What is a TA salary in Sept. 1998?

(d) What is a TA maximum salary under this plan?

9. A charged ion passes between two plates. A varying charge is applied to the two plates, as pictured. Assume the path of the ion is given by the graph of

$$p(x) = (2e^{-x/2})\sin(2x+1),$$

for $0 \leq x \leq 4$.



   (a) Find the locations where the ion is equidistant from the two plates.

   (b) Find the locations where the ion is a distance of 2.2 cm from the bottom plate.

   (c) Suppose the horizontal location $x = x(t) = 200t^2 + 2t$ after t seconds. When is the ion equidistant from the two plates.

   (d) When is the ion closest to the top plate?

10.* Basic atomic theory tells us that Hydrogen H consists of one proton with a single electron orbiting around. A key idea in *quantum mechanics* is that there is NO HOPE of describing the exact location of this electron. So, instead, you introduce a function $p(x)$ that gives the odds the electron is inside a sphere of radius x cm centered at the proton. It is possible to find a formula for $p(x)$ and here it is:

$$p(x) = 1 + \left(-1 - (3.7795 \times 10^8)x - (7.14232 \times 10^{16})x^2\right) e^{-(3.7795\times 10^8)x}.$$

Introduce a constant $a = 5.2917 \times 10^{-9}$ cm; this is called the *Bohr radius*.

   (a) What is the probability the electron is distance at most a from the proton?

   (b) What is the probability the electron is between a cm and $2a$ cm from the proton?

   (c) Use a graphing device to sketch the graph of $y = p(x)$ on the domain $[0, 10a]$. What happens to the graph on the domain $[0, r]$ for $r = 20a, 100a, 1000a$?

   (d) Find a sphere, centered at the proton, such that the electron is inside the sphere exactly 90% of the time.

   (e) Use c. to explain the chances of finding the electron located between 1 cm and 1 meter from the proton.

11. For an animal to live, it must carefully regulate the amount of *glucose* in it's blood supply. (For example, brain function critically depends on glucose level.) One way this is done involves secreting a hormone called *insulin* into the bloodstream. Suppose you are monitoring both the glucose level and insulin level in a cougar. The concentration levels of each substance are given by these two functions:

$$g(t) = 400(e^{-t}\cos(t) + 0.25)\ \text{mg}/100\text{ml} \quad \text{(glucose concentration)}$$
$$i(t) = 400(e^{-t}\sin(t) + 0.25)\ \text{mg}/100\text{ml} \quad \text{(insulin concentration)},$$

where t represents hours since you began monitoring.

(a) Use a graphing device to sketch graphs of $g(t)$ and $i(t)$ in the same coordinate system, for the first 10 hours of observation.

(b) What were the concentrations of glucose and insulin when your observations began?

(c) What was the minimum glucose concentration in the cougar during the first 10 hours of observation?

(d) During the first 10 hours of observation, how often did the insulin level exceed the glucose level?

(e) The normal level for glucose concentration (called a "set point") is 90 mg/100 ml. What is your diagnosis of the condition of this cougar at the end your 10 hours of observation?

12. A model for the United States population (in millions) is given by the function

$$p(t) = \frac{387.9802}{1 + 54.0812e^{-rt}},$$

where r is the *growth rate* and t represents time since the year 1790.

(a) Let $\delta = 0.0270347$. Use a graphing device to sketch the graphs of $p(t)$ for these five values of $r = \frac{1}{2}\delta, \delta, \frac{3}{2}\delta, 2\delta$.

(b) How does changing r affect the resulting predictions of the population model? What is the significance of the number 387.9802?

(c) Assume the growth rate is $r = \delta$. If we vary the growth rate by $\pm 2\%$ (i.e. $r = \delta \pm 0.02\delta$), when should we expect the population of the United States will first exceed 300 million?

13. Nicole has decided to entertain party guests by flipping a coin 10 times. Before she starts, Dave remarks that the graph of the function

$$G(x) = \frac{1}{\sqrt{5\pi}} e^{-\frac{1}{5}(x-5)^2},$$

for $0 \leq x \leq 10$, tells us how to guess the probability she will get exactly k heads in 10 flips. Let's see why.

(a) Plot the graph of $G(x)$ on the given domain of x values, labeling your axes.

(b) The probability of getting exactly k heads in 10 tosses is given by the function

$$H(k) = \frac{10!}{k!(10-k)!}(\frac{1}{2})^{10},$$

for $k = 0, 1, 2, \ldots, 10$ Plot the points $(k, H(k))$, where $k = 0, 1, 2, \ldots, 10$. (Go back and review Exercise 2.1.15.) What is the connection between the plotted points and the graph in a.?