5.4 Loans and Investments

1. Suppose you agree to purchase a house for \$155,000. Assume you make a 10% down payment and the mortgage lender agrees to approve a loan with an annual interest rate of 8.75%.
   (a) If you wish to pay off the house in 15 years with equal monthly payments of d dollars, what is the required monthly payment?
   (b) How much higher is this monthly payment than monthly payment of the 30-year loan?
   (c) Compare the amount of principal and interest paid on the house after one year under the 15 and 30 year loans.
   (d) If you pay off the loan over the full 15 years, how much interest will you pay for this loan compared to the 30 year loan?

2. Suppose you agree to purchase a house for \$155,000. Assume you make a 10% down payment and the mortgage lender agrees to approve a loan with an annual interest rate of 8.75%. If you wish to pay off the house in 30 years with equal monthly payments, we already showed that $d = \$1097.45$.
   (a) If you pay an additional $(\frac{1}{12})d = \$91.45$ toward principal each month, how will this shorten the life of the loan?
   (b) How much will you save in interest using this "extra payment plan" as compared to the full 30 year payment plan? (Hint: Write the formula for the balance if you make monthly payments of $d + (1/12)d$, set equal to 0 and solve for k.)

3. Having just won the Washington State Lottery, you plan to buy a brand new red Ferrari. The dealer will finance the total sales price of \$112,500 at $r = 4.9\%$ annual interest over 5 years. Assume the loan balance is computed as in (5.4.4).
   (a) What is the monthly payment d if the loan is to be paid off in 5 years with equal payments?
   (b) Compute a partial amortization table for the first 4 months of the loan.
   (c) Does the portion of the monthly payment going toward interest ever exceed the portion going toward principal?
   (d) Suppose you make payments for 3 years. Then you want to pay off the balance of the loan; how big is the check you need to write?
   (e) If you pay on the loan for the full 5 years, how much interest will you have paid?
   (f) After how many payments will the total amount of interest paid exceed \$20,000?
   (g) After how many payments is the loan balance under \$10,500?

4. The Mortgage industry uses various rules in order to "qualify a borrower"; i.e. in order to approve a loan. One key rule is a formula which specifies the monthly mortgage payment may not exceed 28% of your gross monthly income (i.e. income before deductions or taxes).
   (a) You have found your dream house, but it requires that you finance \$780,000. If the current 30 year rate is $r = 9\%$, what is your required gross annual income to qualify for the mortgage?
   (b) If you apply to finance \$150,000 for 30 years and your gross annual income is \$40,000, find the largest interest rate under which you will satisfy the 28% rule. (Hint: Begin with (5.4.4) and use a graphing device.)

(c) If you apply to finance \$200,000 for 15 years and your gross annual income is \$80,000, find the largest interest rate under which you will satisfy the 28% rule.(Hint: Use the same trick as b.)

5. You spot a dream vacation property while visiting the San Juan Islands. The property will cost \$267,000 and you plan to make a down payment of 30%. A mortgage broker has offered you a 30 year loan at 7.9% annual interest. Assume the balance of the loan is computed using the formula in (5.4.4).

   (a) If you wish to pay off the loan in equal monthly payments over 30 years, what is the required monthly payment d?

   (b) Write down the loan balance function $B(k)$, inserting the values of d, L, r.

      i. Sketch a graph of this function using your graphing device. Is this a function of exponential type?

      ii. What does the shape of the graph tell you about how fast you are paying off the loan?

   (c) Compute an amortization table for the months $k = 211, 212, 213, 214$ of the loan.

   (d) When will the portion of the monthly payment going toward principal exceed the portion going toward interest? Determine these amounts explicitly.

   (e) When will the portion of the monthly payment going toward principal be at least \$200 more than the portion going toward interest?

   (f) When will the portion of the monthly payment going toward interest be less than \$100?

   (g) Suppose you pay on the loan for 20 years, then pay off the balance in one lump sum. How much interest will you end up paying?

   (h) If you pay on the loan the full 30 years, how much interest will you have paid?

   (i) After how many payments will the balance on the loan be under \$15,000?

   (j) If you pay an extra \$150 each month (i.e. you pay $d+\$150$ each month), when will the loan be paid off?

   (k) You wish to pay an extra amount e dollars each month, in addition to d, so that the loan will be paid off in 15 years. How much is e?

6. Before signing the papers on the vacation home in the previous problem, a competing mortgage broker offers you a 30 year loan at r% annual interest and the balance of the loan is also computed using the formula in (5.4.4). The competing brokers monthly payment to pay off the loan over 30 years of equal payments is \$250 less than the monthly payment in the previous problem. What is the interest rate r? (Hint: You will need a graphing device.)

7. We assume the situation in Example (5.4.7); i.e. you agree to purchase a house for \$155,000. Assume you make a 10% down payment. We have already discussed this loan in the context of the standard loan balance scheme (5.4.4). We now describe an alternate loan balance scheme. Here, a Mortgage company offers an annual interest rate of 8%, but the balance after k payments is computed different from the standard formula. Namely, the borrowed amount L and the monthly payments both grow with **continuous** compound interest (instead of monthly compound interest). If $B^*(k)$ is the balance of the loan after k months, then

$$\begin{aligned} B^*(k) &= amountowedafterkpayments \\ &= amountborrowed + interest - kmonthlypayments + accumulatedinterest \\ &= futurevalueofLdollarsafterkmonthsatr\%continuouscompounding - \\ &\qquad kmonthlypayments + accumulatedcontinuousinterest \end{aligned}$$

We will call this the *continuous loan balance formula.*

(a) Obtain a formula for $B^*(k)$, analogous to the one we obtained in (5.4.4).

(b) Let d^* be the equal monthly payments to pay of the loan in 30 years. What is d^*?

(c) How does the monthly payment d^* under the continuous loan balance plan compare with the monthly payment d in Example (5.4.7) under the standard loan balance plan?

(d) If you pay off the loan over the full 30 years, compare the total interest under the standard and continuous loan balance plans. Which is a better deal for the borrower?

(e) Under the continuous loan balance plan, when will the portion of the monthly payment going toward principal exceed the portion going toward interest? Compute the amounts explicitly.

(f) When will the portion of the monthly payment going toward principal be at least \$200 more than the portion going toward interest?

(g) When will the portion of the monthly payment going toward interest be less than \$100?

(h) If you plan to payoff the loan after 20 years worth of payments, how much will you owe?

(i) After how many payments will the balance on the loan be under \$15,000?

(j) If you pay an extra \$250 each month (i.e. you pay $d + \$250$ each month), when will the loan be paid off?

(k) You wish to pay an extra amount e^* dollars each month, in addition to d^*, so that the loan will be paid off in 15 years. How much is e^*?

8. We continue with the situation in Example (5.4.7); i.e. you agree to purchase a house for \$155,000. Assume you make a 10% down payment. One lender offers you 8.75% interest for a 30 year loan using the standard loan balance formula in (5.4.4) and having equal monthly payments of d. A second lender offers you a 30 year loan at r% interest using the continuous balance formula of the previous problem and has equal monthly payments d^*.

   (a) If $d = d^*$, what is r?

   (b) If you are shopping to find a continuous loan balance lender with $d^* = \$500$, what is the required rate r?

9. The *EMS Loan Company* advertises a car loan plan which computes the loan balance using the formula in (5.4.4) and an annual interest rate of 8.4%. You intend to borrow \$34,000 and pay the loan off in 5 years with equal monthly payments.

   (a) What are the equal monthly payments d under the *EMS* loan scheme?

   (b) What is the balance of the loan after one year using the monthly payments d in part a.?

   (c) When will the *EMS* loan be paid off if you pay an extra \$150 each month?

10. At age 32 you open an IRA with \$2000 and add \$750 per month. Assume an annual rate of 6% and use the balance formula in (5.4.11).

    (a) What is the value of the account at age 60?

    (b) When will the account contain \$500,000?

    (c) If you stop monthly contributions at age 40, then transfer the balance into a continuous compounding account earning 3.4% annual interest, when will the account value reach \$500,000?

11. At age 40 you open an IRA with \$500 and make monthly contributions of d. Assume a constant annual interest rate of $r\%$ and use the future balance formula in (5.4.11). Your goal is to retire at age 60 with \$1,000,000 in the account.

   (a) If $r = 8\%$, what is the required monthly payment to achieve your goal?

   (b) If the monthly contribution is $d = 1200$, what is the required interest rate to achieve your goal? (Hint: You will need a graphing device.)

12. The *Upfront Investments Co.* offers an Investment account which has a future balance $V^*(k)$ calculated according to a scheme different from the one leading to (5.4.11). Assume you begin with L and an annual interest rate of $r\%$. This firm will pay the interest on each monthly contribution d "up front"; e.g. the value of the account after each of the first three payments will be

$$\begin{aligned} V^*(1) &= L(1+\frac{r}{12}) + d(1+\frac{r}{12}), \\ V^*(2) &= L(1+\frac{r}{12})^2 + d(1+\frac{r}{12}) + d(1+\frac{r}{12})^2, \\ V^*(3) &= L(1+\frac{r}{12})^3 + d(1+\frac{r}{12}) + d(1+\frac{r}{12})^2 + d(1+\frac{r}{12})^3, \textit{etc.} \end{aligned}$$

   (a) Find the formula for $V^*(k)$.

   (b) If $L = \$1000$, $r = 7\%$ and $d = \$50$, contrast the future value of the Upfront account versus an an account using the formula in (5.4.11). After 30 years, what would be the difference?

13. At age 30 you open an Investment account with \$2000, make monthly contributions of \$1000 and the annual rate of interest is $r = 7.5\%$. The future value $V(k)$ of this account after k months is given by the formula in (5.4.11). At age 40, you sign the papers on a 15 year home Mortgage. The Mortgage is for \$180,000 with an interest rate of $r = 9\%$ and the loan balance after k payments is computed using the formula in (5.4.4). Let d be the equal monthly mortgage payments. The Mortgage lender has arranged to have your monthly payment d subtracted from your Investment account each month for 15 years. You continue to make the \$1000 monthly investment contributions.

   (a) What is the balance in the Investment account when you sign the Mortgage papers?

   (b) What is the balance in the Investment account when you are age 45?

   (c) When the balance of the Mortgage is \$150,000, how much money is in the Investment account?

   (d) After 15 years of Mortgage payments, does the Investment account have positive or negative value. If a negative value, at what point did the Investment account exhaust? If positive, what is the final balance in the Investment account when the house is paid off?

14. (a) Investor A opens an IRA at age 30 with \$2000 and contributes \$200 each month for 10 years. At age 40 the investor ceases monthly contributions to the IRA. During the ten year period from age 30 to 40, the IRA future value is computed using (5.4.11) and $r = 6\%$. After age 40, the account grows according to monthly compounding with the same interest rate. How much is in the account when the investor reaches age 65? What is the investor's capital gain; i.e. the account value minus funds the investor contributed.

   (b) Investor B opens an IRA at age 40 with \$2000 and contributes \$200 each month. The IRA future value is computed using (5.4.11) and $r = 6\%$. How much is in the account when the investor reaches age 65? What is the investor's capital gain; i.e. the account value minus funds the investor contributed?

15. Suppose you are a Professor at the University of Washington. And, further suppose that your tenure at UW will be 32 years, your starting salary is \$40,000 per year, and your salary increases *linearly* so that in your last year your salary is 2.5 times your starting salary. You retire at 65 years old, so you started your tenure when you were 33 years old.

   (a) Write the formula $S = f(t)$ giving your salary as a function of t, where $t = 1$ corresponds to the first year of your employment.

   (b) You want to create an investment strategy that will give you a comfortable retirement. So you decide to invest 5% of your 32 year average annual salary in an Individual Retirement Account (IRA). What is your annual investment in the IRA?

   (c) The investment firm that manages your money has a long term history of returning at least 10% interest per year after management fees. You start your IRA with $L = \$2000$ and use formula (5.4.11) to compute the future value of the account. What is your investment balance when you retire?

   (d) What was your total investment in the IRA?

   (e) How much interest did you earn from the IRA?

   (f) Now, you have retired. You move your money from your IRA account into an annuity account that pays 12% interest on your balance. The annuity balance is the same as a loan balance given by:

$$A(k) = L\left(1 + \frac{r}{12}\right)^k - \frac{12d}{r}\left(\left(1 + \frac{r}{12}\right)^k - 1\right)$$

   Here $L =$"the final balance from your IRA account"; $r =$"interest earned during the life of your annuity"; $k =$"months of annuity payments"; and, $A(k)$ is now the balance in your annuity account.

   (g) Suppose you want to have a balance of $A(k) = 0$ by the time you are 85 years old. What are your monthly payments over the life of your annuity?

16. Suppose you agree to purchase a house for \$155,000. Assume you make a 10% down payment and the mortgage lender agrees to approve a loan with an annual interest rate of 8.75%. We are going to graphically compare various monthly payment plans, always using the loan balance formula (5.4.4).

   (a) If you make monthly payments of $d = \$1097.45$, $d = \$1188.90$ or $d = \$1050$, use a graphing device to simultaneously sketch the three functions representing the loan balances after k months. From your standpoint as the person paying the mortgage, interpret this picture.

   (b) Compare the loan balance graphs if you are making a monthly payment of d=\$1017 versus d=\$1018. Something very interesting is happening; explain.