5.3 Logarithmic Functions

1. (a) Compute: $\log_5 3, \log_e 11, \log_{\sqrt{2}} \pi, \log_2 10, \log_{10} 2$.
   (b) Solve for x: $35 = e^x, \log_3 x = e, \log_3 5 = xe^3$.
   (c) Solve each of these equations for x in terms of y: $y = 10^x, 3y = 10^x, y = 10^{3x}$.

2. For each of the following, which is the correct answer:
   (a) $\ln(\frac{x^3}{y^2}) =$
   (i) $(\ln(x))^3 - (\ln(y))^2$ $\quad (ii) \frac{(3\ln(x))}{2\ln(y)}$ $\quad (iii) 3\ln(x) - 2\ln(y)$ $\quad (iv) \frac{(\ln(x))^3}{(\ln(y))^2}$.
   (b) $2\ln(x) - \ln(x-3) =$
   $(i) \ln(\frac{x^2}{x-3})$ $\quad (ii) \ln(x^2 - x + 3)$ $\quad (iii) \ln(\frac{2x}{x-3})$ $\quad (iv) \frac{\ln(x^2)}{\ln(x-3)}$
   (c) $\frac{\ln(x^3)+\ln(x^2)}{\ln(x)} =$
   $(i) \ln(x^4)$ $\quad (ii) \ln(x^2 + x)$ $\quad (iii) 5$ $\quad (iv) \ln(\frac{e^{x^3}+e^{x^2}}{e^x})$
   (d) If $\log_{10}(y) = 3x\log_{10}(2)$, then $y =$
   $(i) 2(10^{3x})$ $\quad (ii) 2^{3x}$ $\quad (iii) 10^{8x}$ $\quad (iv) 10^{3x} + 2$

3. Solve the following equations for x:
   (a) $\log_3(5) = \log_2(x)$
   (b) $10^{\log_2(x)} = 3$
   (c) $3^{5^x} = 7$
   (d) $\log_2(\ln(x)) = 3$
   (e) $e^x = 10^5$
   (f) $2^{3x+5} = 3^2$
   (g) $e^{\sin(x)} = \frac{1}{2}$
   (h) $e^{\tan(x)} = 2$
   (i) $\sin(e^x) = \frac{1}{2}$

   Solve the following equations for x. In each case, simultaneously graph each side of the equation on your graphing device, then interpret your solutions.
   (j) $2^x = e^{3x+1}$
   (k) $e^{x^2+2x-1} = 0.5^x$
   (l) $2^{3x-4} = 3^{x^2+x-3}$
   (m) $10\sin(e^{-x^2+1}) = 6$
   (n) $e^{2x} = e^{-x}$
   (o) $e^{x^2+x+2} = 10$
   (p) $e^{x^2+x+2} = 5$
   (q) $2^{(\ln(x^2+1))^2+1} = 3$

4. Return to the exponential models for Men's and Women's Earning Power in Exercise 5.2.9: $M(x) = 9521(1.0662066)^x$ and $W(x) = 5616(1.0727495)^x$. Determine when the earning power of women will exceed the earning power of men.

5. (a) If \$250,000 is invested in a continuous compounding savings account and we want the value after 25 years to be \$2,300,000, what is the required annual interest rate?
   (b) If \$250,000 is invested in a continuous compounding savings account with $r = 6.4\%$ annual interest, when will the exact account value be be \$2,300,000?
   (c) If \$250,000 is invested in a monthly compounding savings account and we want the value after 25 years to be \$2,300,000, what is the required annual interest rate?

(d) If \$250,000 is invested in a monthly compounding savings account with $r = 6.4\%$ annual interest, when will the exact account value be be \$2,300,000?

6. A gas filled balloon is released and begins floating upward into the sky. Gas is escaping from the balloon so that the volume is decaying exponentially. The volume is given by $V(t) = Ab^{-t}$, where t represents seconds elapsed since release, A is a positive constant and $b > 0$. The volume of the balloon is 5 liters when released and 0.5 liters after 4 seconds. The balloon altitude is $a(t) = 10\sqrt{t}$ meters t seconds after release.

   (a) Calculate the values for A and b.

   (b) How much air is left in the balloon 2 seconds after release?

   (c) When the volume of the balloon is 0.001 liters, what is the altitude of the balloon?

   (d) When the balloon reaches an altitude of 100 meters, what is the volume of the balloon?

7. *Big Promise Investments* advertises a special investment account. According to the prospectus, if you invest B dollars, the account value after t years is computed using three criteria: First, for less than 5 years, the account value is computed by using yearly compounding and an annual interest rate of $r = 4.45\%$. For exactly 5 years, the account value is computed by using continuous compounding and an annual interest rate of $r = 5.75\%$. For more than 5 years, the account value is computed by applying quarterly compounding to the 5 year account value for $t - 5$ years at an annual rate of $r = 9.5\%$. The minimum investment in the account is \$2000.

   (a) Given an initial investment of \$2000, when will the account value equal \$2173?

   (b) Given an initial investment of \$2000, what is the account value after 5 years?

   (c) Given an initial investment of \$2000, when will the account value equal \$2650?

   (d) When will an initial investment of B dollars increase by 6.2%?

   (e) When will an initial investment of B dollars triple in value?

8. In 1883, your great grandfather purchased a piece of property in downtown Philadelphia which was then valued at \$1,200. You recently discovered the Deed for this property and have contacted the City of Philadelphia concerning the current value of the property. The reply you receive involves good news and bad news. First, you are told that the value of the property t years after it was purchased is computed using monthly compounding and an annual interest rate of 4%. You are also told that your great grandfather forgot to pay taxes on the property the year it was purchased, which amounted to 1% of the value. The City informs you that the penalty on the never paid \$12 carries an annual interest rate of 8% continuously compounded.

   (a) When is the property valued at \$ 1,000,000?

   (b) When is the back tax equal to \$1,000,000?

   (c) When will the back tax equal the value of the property?

   (d) If you sell the property in 1996, will you realize a profit?

9. While attending the Chicago Symphony Orchestra performance of the Stravinsky *Firebird*, you carry along a portable sound pressure meter. Your meter measures a sound pressure level of 94 db at the ending Finale. Earlier, during the quietest passage of the piece, when only the string section is playing, your meter measures a sound pressure level of 18 db.

   (a) Find the ratio of the sound intensity of the Finale and quietest passage.

(b) If the CSO plays the Finale as an encore and your meter now measures a maximum sound pressure reading of 95 db, by what factor has the intensity of the sound increased from the first to second reading?

(c) If a passage produces a sound intensity one-fifth that of the Finale, what is the sound pressure level?

10. The *Richter Scale* is used to measure the strength of an earthquake. The typical unit is $R = \log_{10}(\frac{I}{I_o})$, where I_o is the minimal intensity one can detect and I is the quake intensity.

(a) Discuss the ratio of the intensity between earthquakes measuring 6 and 7 on the Richter scale.

(b) Suppose an earthquake is measured 8.2 on the Richter scale and an aftershock is one-third the intensity. What is the measurement of the aftershock on the Richter scale?

11. A boat in the middle of the ocean contains a population of ants and their population after t hours is $P(t) = 2(3^t)$. Once there are 4,000,000 ants it will become unpleasant. It will take 12 hours for you to reach shore. Will you reach shore before unpleasantness sets in?

12. In 1987, the population of Mexico was estimated at 82 million people, with an annual growth rate of 2.5%. The 1987 population of the United States was estimated at 244 million with an annual growth rate of 0.7 %. Assume the future population of each country is predicted by the "continuous compounding" formula in (5.2.6).

(a) When will Mexico double its 1987 population?

(b) When will the United States and Mexico have the same population?

13. (a) If you invest P_o dollars at 7% annual interest and the future value is computed by continuous compounding, how long will it take for your money to double?

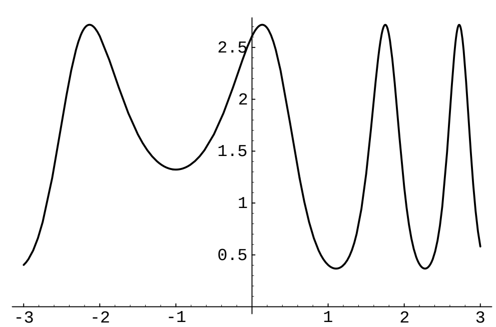
(b) Suppose you invest P_o dollars at r% annual interest and the future value is computed by continuous compounding. If you want the value of the account to double in 2 years, what is the required interest rate?

(c) A rule of thumb used by many people to determine the length of time to double an investment is the *rule of 70*. The rule says it takes about $t = \frac{70}{r}$ years to double the investment. Graphically compare this rule to the one isolated in part b. of this problem.

14.* Consider the function

$$y = e^{\sin(x^2+2x-5)}$$

graphed at right. Solve the equation $2 = f(x)$ on the domain of x-values [-2,2]. Interpret these solutions graphically.



15. (a) Return to Exercise 5.2.7. Sketch the graph of $y = \cosh(x)$ and $y = \cosh^{-1}(x)$ in the same coordinate system; discuss the two cases $x \geq 0$ and $x \leq 0$ separately.

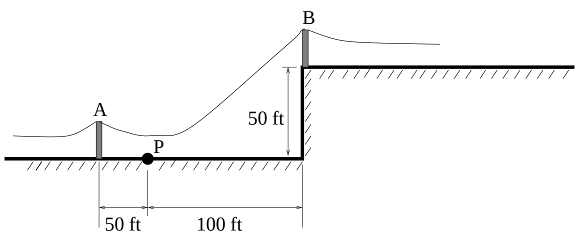
(b) Obtain a "formula" for $y = \cosh^{-1}(x)$. (Hint: First write out this equation, then try to solve for x in terms of y. To do this, make the substitution $t = e^x$ and first solve for t in terms of y.)

(c) Return to Exercise 5.2.8 and determine where the cable is 85 feet above the ground.

16. Recall exercise 5.2.7. *Potomac Power Co.* needs to install a new powerline from the top of a cliff to the level ground below, as pictured. If we impose a coordinate system so that the origin is at the base of pole "A", the hanging cable is modeled by a portion of the graph of

$$f(x) \quad = \quad 100\cosh\left(\frac{x-50}{100}\right) - 80.$$

At the position labeled "P", the powerline is exactly 20 feet above the ground, which is the minimum distance from the ground to the cable.



(a) Find the height of poles "A" and "B".

(b) What portion of the powerline is at least 25 feet above the level ground?

(c) What portion of the cable is above the level of the cliff top?

(d) What is the horizontal distance from the base of pole "B" to the cable?

17. A particle is moving in the xy-plane along the right-hand branch of the unit hyperbola $x^2 - y^2 = 1$. The unit of measure used on each axis will be inches. The particle location after t seconds is the point $P(t) = (\cosh(3t-1), \sinh(3t-1))$.

(a) Where is the particle located at time $t = 0$ and time $t = 1$?

(b) Sketch a picture of the situation for the first second, indicating the direction of motion along the hyperbola.

(c) When will the particle cross the x-axis?

(d) When will the particle have x-coordinate 1.2?

(e) When will the particle have y-coordinate ± 1? (Hint: You will need to mimic the arguments used in Exercise 5.2.12 and determine the inverse hyperbolic sine function $\sinh^{-1}$.)

(f) When will the particle be 2 inches from the origin?

18. Your Grandfather purchased a house for \$55,000 in 1952 and it has increased in value according to a function $y = v(x)$, where x is the number of years owned. These questions probe the future value of the house under various mathematical models.

(a) Suppose the value of the house is \$75,000 in 1962. Assume $v(x)$ is a linear function. Find a formula for $v(x)$. What is the value of the house in 1995? When will the house be valued at \$200,000?

(b) Suppose the value of the house is \$75,000 in 1962 and \$120,000 in 1967. Assume $v(x)$ is a quadratic function. Find a formula for $v(x)$. What is the value of the house in 1995? When will the house be valued at \$200,000?

(c) Suppose the value of the house is \$75,000 in 1962. Assume $v(x)$ is a function of exponential type. Find a formula for $v(x)$. What is the value of the house in 1995? When will the house be valued at \$200,000?

19. As light from the surface penetrates water, its intensity is diminished. In the clear waters of the Caribbean, the intensity is decreased by 15 percent for every 3 meters of depth. Thus, the intensity will have the form of a general exponential function.

    (a) If the intensity of light at the water's surface is I_o, find a formula for $I(d)$, the intensity of light at a depth of d meters. Your formula should depend on I_o and d.

    (b) At what depth will the light intensity be decreased to 1% of its surface intensity?

20. The average tenure of a Professor at the University of Washington is 31.6 years. The administration believes that a Professor's salary after 31.6 years of service should be 2.5 times his/her "hiring in" salary. Assume the Professor's salary grows with continuous compounding according to this constraint.

    (a) What is the annual rate r of salary growth?

    (b) Assume inflation grows at an annual rate of $r = 3.4\%$, compounded continuously. If a Professor is hired at \$30,000, what is the inflation adjusted buying power of his/her salary at retirement after 40 years of service?

21. *MacForever* Magazine has published data on the average downtime due to system crashes for the new Windows 95 PC operating system. One month after installation, the user average downtime was 2%. Six months after installation, the user average downtime was 3.2%. The magazine goes on to claim "...downtime is increasing exponentially...". If this is the case, find an exponential model $d(t)$ which predicts the average amount of downtime t months after installation of the Windows 95 operating system. What is the downtime two years after installation? When will the computer be down 100% of the time?

22. Return to Exercise 5.2.10. When will the population of Pinedale double under the linear model and under the exponential model (compared to the population of 860 in 1990)? When will it triple (compared to the population of 860 in 1990)?

23. During Autumn term, the *Happiness Factor* in a Chemistry class on day x of the term is calculated to be

    $h(x)$= 100(number happy students on day x)/ (total enrolled students on day x);

    i.e. $h(x)$ tells us the percentage of happy students on day x of the term.

    On the 5^{th} day of the term the happiness factor was 80%, while on the 25^{th} day of the term the happiness factor was 60%. Assume the function $h(x)$ is of exponential type.

    (a) Find a formula for $h(x)$.

    (b) What was the happiness factor on day 75, the last day of the term?

    (c) When is the happiness factor 40%?

    (d) Assume the percentage of Chemistry Majors in the same class on day x is given by the function $c(x) = 65e^{-0.01x}$. When will the percentage of Chemistry Majors exceed the percentage of happy people in the class?

24. When a switch is thrown in a certain electrical circuit, the amount of current I (in amps) flowing at time t (in seconds) is given by

$$I(t) = 65(1 - e^{-2t}).$$

(a) What is the current the instant the switch is thrown?

(b) What is the current after 1.2 seconds? What is the current after 3 minutes?

(c) How long is the current between 15 and 20 amps.

(d) What is the practical significance of the number "65" in the formula for $I(t)$?

25. The length of some fish are modeled by a *von Bertalanffy* growth function. For Pacific halibut, this function has the form

$$L(t) = 200\,(1 - 0.956\,e^{-0.18t})$$

where $L(t)$ is the length (in centimeters) of a fish t years old.

(a) What is the length of a new-born halibut at birth?

(b) Use the formula to estimate the length of a 6–year–old halibut.

(c) At what age would you expect the halibut to be 120 cm long?

(d) What is the practical (physical) significance of the number 200 in the formula for $L(t)$?

26. Rewrite each function in the form $y = A_o e^{at}$, for appropriate constants A_o and a.

(a) $y = 13(3^t)$

(b) $y = -7(1.567)^{t-3}$

c. $y = -17(2.005)^{-t}$

d. $y = 3(14.24)^{4t}$

27. Complete the table:

$P(t) = P_o e^{rt}$	$P(t) = P_o b^t$	$P(0)$	point on $y = P(t)$	point on $y = P(t)$
$P(t) = 5000e^{0.03t}$			$(1, \quad)$	$(\quad , 40034)$
	$P(t) = 84(1/4)^t$		$(-3, \quad)$	$(\quad , 2)$
		34	$(7, 120)$	$(\quad , 1)$
			$(4, 20)$	$(19, 3)$

28. A *cancerous cell* lacks normal biological growth regulation and can divide continuously. Suppose a single mouse skin cell is cancerous and its *mitotic cell cycle* (the time for the cell to divide once) is 20 hours. The number of cells at time t grows according to an exponential model.

(a) Find a formula $C(t)$ for the number of cancerous skin cells after t hours.

(b) Assume a typical mouse skin cell is spherical of radius 50×10^{-4} cm. Find the combined volume of all cancerous skin cells after t hours. When will the volume of cancerous cells be 1 cm^3?

29. Ever wonder how they get enough DNA from a crime scene to do any analysis? Here is the way it works. If you have just one piece of a DNA molecule, you can make a large number of exact copies of a particular segment using a device called a *thermal cycler*. You put your DNA sample into the device, along with some extra ingredients. Then every c minutes it completes one "cycle"; c is called the *cycle time* and depends on the size of the segment you are duplicating. After 1 cycle there are 2 copies of the DNA segment; after 2 cycles there are 4 copies; etc. This process continues, doubling the number of exact copies of the desired DNA segment with each cycle.

(a) What is the formula for the number of copies of the DNA segment after n cycles? How many copies will you have after 10 cycles, 20 cycles, 30 cycles?

(b) If the cycle time is $c = 3.7$ minutes, when you will have 10^{10} copies.

30. A geneticist has a collection of yeast cells of various ages on a culture dish. The ages of these cells vary from $t = 0$ days (meaning the cell just arose via cell division) to age $t = 90$ days. She has tabulated data on the death rate per 1000 cells. Assume that the death rate per 1000 cells is given by a function $y = y(t)$ of exponential type; i.e. $y = Ae^{rt}$, for some A, r. Our goal is to find a "best fit" choice for A, r.

age t days	death rate y per 1000 cells
0	0.3
20	1.9
40	2.9
80	83.5
90	181.9

(a) Given the first row of the table, compute A.

(b) Sketch a ty-coordinate system, where t represents days. Plot the given data. Does it seem reasonable to "fit" this data with a function of exponential type?

(c) Sketch a *new* $t, \ln(y)$-coordinate system and plot the points $(t, \ln(y))$ corresponding to our data. (The vertical axis is the $\ln(y)$ axis.) What sort of curve would best fit this data?

(d) If we take the natural log of each side of the exponential model, we get $\ln(y) = \ln(Ae^{rt}) = \ln(A) + rt$. Using the value of A from a., sketch a graph of this line in the coordinate system of c. which seems to reasonably fit the data. (You can't be exact here, since we don't yet know r, but you can draw a reasonable line. Our goal is to find the correct r.)

(e) Explain the significance of the equation $E = E(r)$ below in terms of your work in d.:

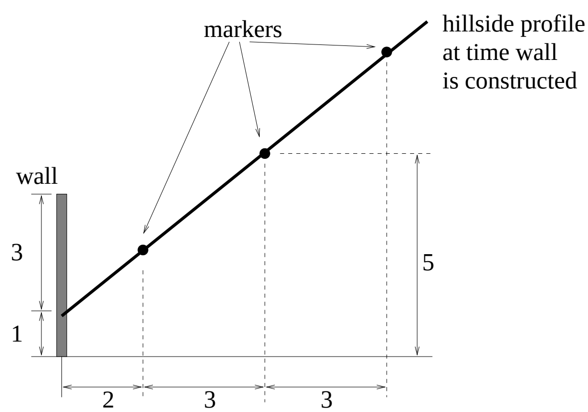
$$\begin{aligned} E &= E(r) = [\ln(1.9) - (\ln(0.3) + 20r)]^2 + [\ln(2.9) - (\ln(0.3) + 40r)]^2 + \\ &\quad +[\ln(83.5) - (\ln(0.3) + 80r)]^2 + [\ln(181.9) - (\ln(0.3) + 90r)]^2 \end{aligned}$$

(f) Using a graphing device, plot the function $E(r)$. This function will have a minimum value; why do we know that without even having to draw the graph? What are the coordinates of the lowest point on the graph?

(g) Use f. to determine the value of r so that the model $y = Ae^{rt}$ best fits our data.

31. A contractor has just built a retaining wall to hold back a sloping hillside. To monitor the movement of the slope the contractor places marker posts at the positions indicated in the picture; all dimensions are taken in units of meters. Assume that the hillside moves as time goes by and the hillside profile is modeled by a function $g_n(x)$ after n years. In Exercise 2.4.9, we showed

$$y = g_n(x) = (0.8)^{n+1}x + \frac{1 - 0.8^{n+1}}{1 - 0.8}.$$



Find when the hillside first starts to spill over the retaining wall.