3.6 Inverse Circular Functions

1. Let's make sure we can handle the symbolic and mechanical aspects of working with the inverse trigonometric functions:

   (a) Set your calculator to "radian mode" and compute to four decimal places:

      i. $\sin^{-1}(x)$, for $x = 0, 1, -1, \frac{\sqrt{3}}{2}, 0.657, \frac{-3}{11}, 2.$

      ii. $\cos^{-1}(x)$, for $x = 0, 1, -1, \frac{\sqrt{3}}{2}, 0.657, \frac{-3}{11}, 2.$

      iii. $\tan^{-1}(x)$, for $x = 0, 1, -1, \frac{\sqrt{3}}{2}, 0.657, \frac{-3}{11}, 2.$

   (b) Redo part (a) with your calculator set in "degree mode".

   (c) Find four solutions of $\tan(2x^2 + x - 1) = 5$. Use a graphing device to interpret your four solutions.

   (d) Solve for x: $\tan^{-1}(2x^2 + x - 1) = 0.5$

2. (a) If $y = \sin(x)$ on the domain $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, what is the domain D and range R of $y = 2\sin(3x - 1) + 3$? How many solutions does the equation $4 = 2\sin(3x - 1) + 3$ have on the domain D and what are they?

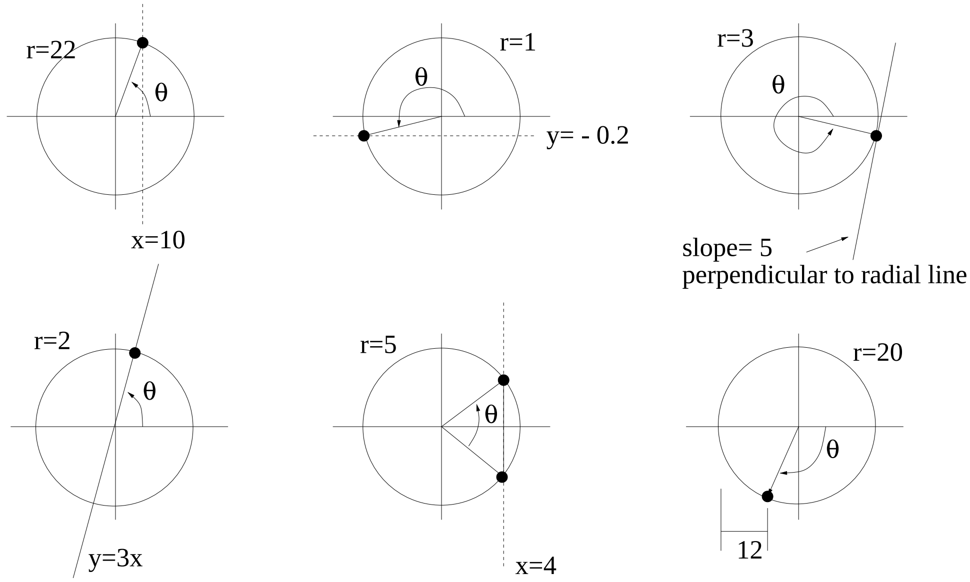
   (b) If $y = \sin(t)$ on the domain $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$, what is the domain D and range R of $y = 8\sin(\frac{2\pi}{1.2}(t - 0.3)) + 18$. How many solutions does the equation $22 = 8\sin(\frac{2\pi}{1.2}(t - 0.3)) + 18$ have on the domain D and what are they?

   (c) If $y = \sin(t)$ on the domain $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$, what is the domain D and range R of $y = 27\sin(\frac{2\pi}{366}(t - 80.5)) + 45$. How many solutions does the equation $40 = 27\sin(\frac{2\pi}{366}(t - 80.5)) + 45$ have on the domain D and what are they?

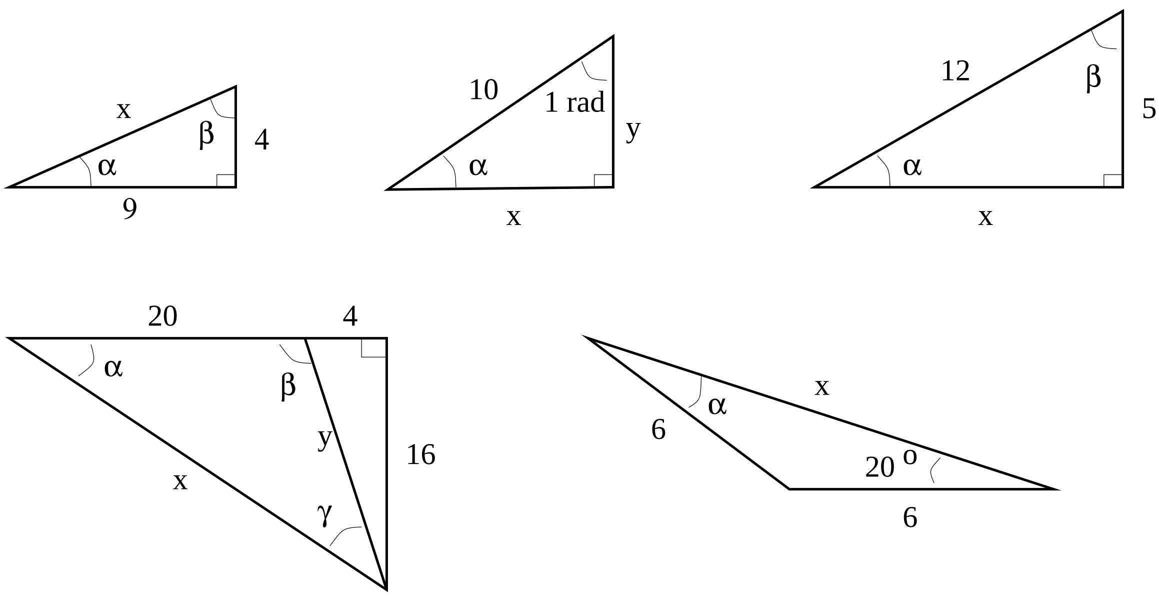
   (d) If $y = \cos(x)$ on the domain $0 \leq x \leq \pi$, what is the domain D and range R of $y = 4\cos(2x + 1) - 3$? How many solutions does the equation $-1 = 4\cos(2x + 1) - 3$ have on the domain D and what are they?

   (e) If $y = \tan(x)$ on the domain $-\frac{\pi}{2} < x < \frac{\pi}{2}$, what is the domain D and range R of $y = 2\tan(-x + 5) + 13$? How many solutions does the equation $100 = 2\tan(-x + 5) + 13$ have on the domain D and what are they?

3. Find the angle θ in each of the pictures below:



4. Find the unknown sides and angles in these triangles:



5. For each of the functions below:

A. Sketch the function and show the horizontal line where $y =$ constant.

B. Find the principal solution and the symmetry solution.

C. Indicate the other solutions on your graph and describe their relationship to the principal and symmetry solutions.

(a) $y = \sin\left(x - \frac{\pi}{2}\right)$, $y = \frac{1}{3}$.

(b) $y = \sin\left(x + \frac{\pi}{6}\right)$, $y = -1$.

(c) $y = \sin(2x - \pi)$, $y = \frac{1}{4}$.

(d) $y = -2\sin(2x - \pi) + 3$, $y = \frac{3}{2}$.

(e) $y = \cos(2x - \pi) + 1$, $y = \frac{4}{3}$.

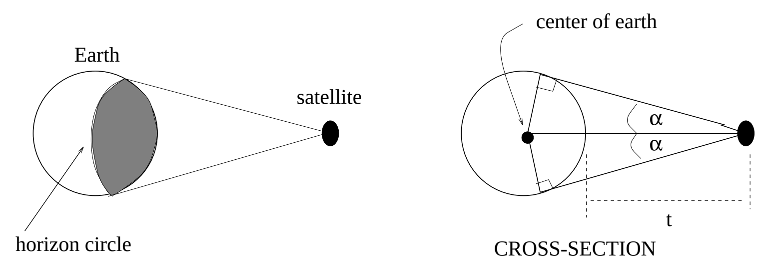
(f) $y = 10\cos(2x + 1) - 5$, $y = -1$.

6. (a) Write down a complete english sentence explaining what this equation tells us: $17.4576^o = \sin^{-1}(0.3)$.

   (b) Write down a complete english sentence explaining what this equation tells us: 0.8632 rads $= \cos^{-1}(0.65)$.

   (c) Write down a complete english sentence explaining what this equation tells us: 1.5529 rads $= \tan^{-1}(56)$.

7. A communications satellite orbits the earth t miles above the surface. Assume the radius of the earth is 3960 miles. The satellite can only "see" a portion of the earth's surface, bounded by what is called a *horizon circle*. This leads to a two-dimensional cross-sectional picture we can use to study the size of the horizon slice:



   (a) Find a formula for α in terms of t.

   (b) If $t = 30,000$ miles, what is alpha? What percentage of the circumference of the earth is covered by the satellite? What would be the minimum number of such satellites required to cover the circumference?

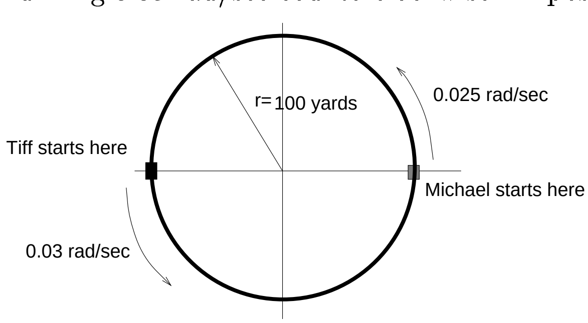
   (c) If $t = 1,000$ miles, what is alpha? What percentage of the circumference of the earth is covered by the satellite? What would be the minimum number of such satellites required to cover the circumference?

   (d) Suppose you wish to place a satellite into orbit so that 20% of the circumference is covered by the satellite. What is the required distance t?

8. The voltage E of a circuit after t seconds ($t > 0$) is given by the equation $E = 120\cos(20\pi t)$; angles are in radian measure. Find the times during the first 1 second when $E = 10$.

9. Solve the equation $2\sin^2(\theta) = 1 - \sin(\theta)$, if we restrict $0 \le \theta \le 2\pi$. Do this by making the initial substitution $z = \sin(\theta)$. If we assume θ is an acute angle, how many solutions does the equation have? What is the general solution?

10. Tiffany and Michael begin running around a circular track of radius 100 yards. They start at the locations pictured. Michael is running 0.025 rad/sec counterclockwise and Tiffany is running 0.03 rad/sec counterclockwise. Impose coordinates as pictured.



   (a) Where is each runner located (in xy-coordinates) after 8 seconds?

(b) How far has each runner traveled after 8 seconds?

(c) Find the angle swept out by Michael after t seconds.

(d) Find the angle swept out by Tiffany after t seconds.

(e) Find the xy-coordinates of Michael and Tiffany after t seconds.

(f) Find the first time when Michael's x-coordinate is -50.

(g) Find the first time when Tiffany's x-coordinate is -50.

(h) Find when Tiffany passes Michael the first time.

(i) Find where Tiffany passes Michael the first time.

(j) Find when Tiffany passes Michael the second time.

(k) Find where Tiffany passes Michael the second time.

(l) Find when Tiffany and Michael have the same x coordinate the first time.

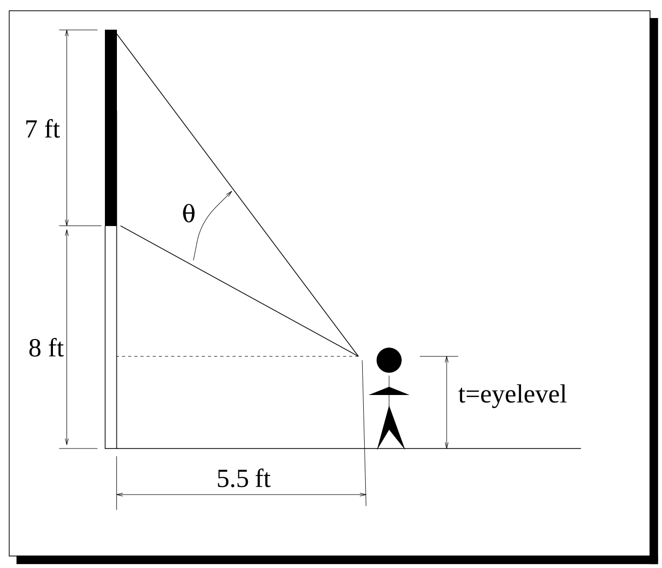
(m) Find when Tiffany and Michael have the same y coordinate the first time.

11. A store wishes to place a large advertising display near its main entrance. The display is a rectangular sign 7 feet high that is mounted on a wall such that the bottom edge is 8 feet from the floor. An observer standing 5.5 feet from the wall with an eye level t feet above the ground has an indicated viewing angle of θ.

(a) Find θ when $t = 5$ feet.

(b) Assuming $0 \leq t \leq 6$, what is the range of possible viewing angles?

(c) What eye level is required if the best viewing angle occurs when $\theta = 22^o$? (Hint: You will need to use an identity: $\tan(\alpha + \beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$.)



12. (a) Return to the Pocket Billiard example in §3.3.1. We already showed that you must strike the cushion $x = 3.75$ feet above the lower left corner pocket. Find the angle θ the path makes with the cushion?

(b) Return to Exercise 3.3.14. Find the angles the billiard ball path makes with the left and bottom cushions.

13. With your calculator:

(a) Compute $\cos(\frac{3\pi}{2})$ and $\cos^{-1}(cos(\frac{3\pi}{2}))$. Explain what has happened.

(b) Compute $\cos^{-1}(2)$. Explain what has happened.

14. Assume that the number of hours of daylight in New Orleans in 1994 is given by the function $D(x) = \frac{7}{3}\sin\left(\frac{2\pi}{365}x\right) + \frac{35}{3}$, where x represents the number of days after March 21.

(a) Find the number of hours of daylight on January 1, May 18 and October 5.

(b) On what days of the year will there be approximately 10 hours of daylight?

15. The voltage output of a circuit at time t seconds is described by the function $v(t) = \frac{1}{1+3sin^2(2t+1)}$.

    (a) What is the voltage at time $t = 0$?

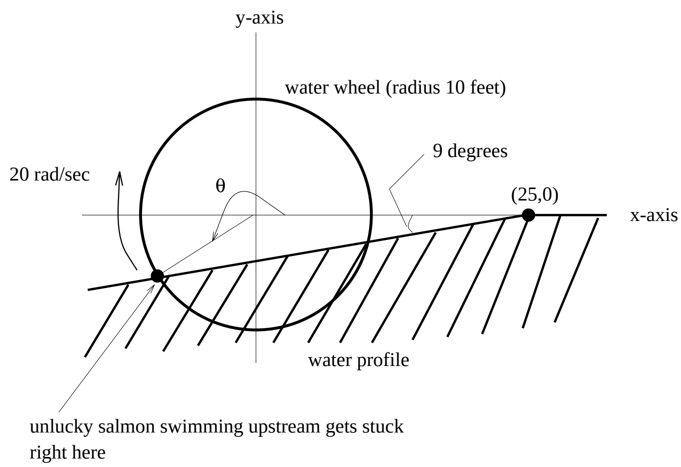
    (b) What is the largest and smallest possible voltage output of the circuit?

    (c) During the first 4 seconds, when will the output voltage be 1 volt?

    (d) During the first 4 seconds, when will the output voltage be at least 0.5 volts?

16. The air temperature (F^o) above a frozen lake during the month of June is given by $L(t) = 20\sin(\frac{\pi}{12}t - \frac{5\pi}{6}) + 45$, where t represents hours elapsed since 12:00am June 1. The lake ice will begin to break up after 100 hours of 60^o thaw time; you can only count time periods when the temperature is at least 60^o. When will the ice break up?

17. Flowing water causes a water wheel to turn in a CLOCKWISE direction. The wheel has radius 10 feet and has an angular speed of $\omega = -20$ rad/sec. Impose a coordinate system with the center of the wheel as the origin. The profile of the flowing water is pictured below. At time $t = 0$, an unlucky salmon gets caught in the wheel at the pictured location. Distance units will be "feet" and time units will be "seconds".



    (a) How long does it take the salmon to complete one revolution?

    (b) What is the salmon's linear speed?

    (c) What is the equation of the line modeling the profile of the flowing water?

    (d) Where (xy-coordinates) does the salmon get stuck on the wheel?

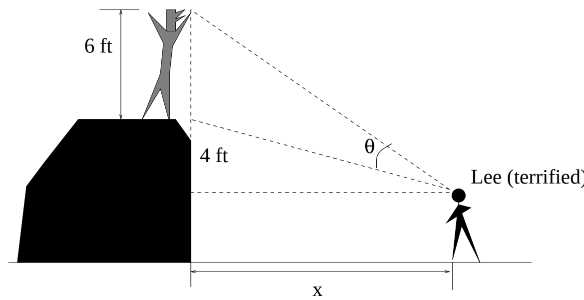
    (e) What is the initial angle θ pictured?

    (f) Let $P(t) = (x(t), y(t))$ be the location of the salmon at time t seconds; find the formulas for $x(t)$ and $y(t)$.

    (g) Where (xy-coordinates) is the salmon located after 0.12 seconds?

    (h) When is the salmon first 4 feet to the right of the y-axis?

18.* Lee is on a tour through the *Olympic Game Farm* and spots an escaped cougar standing on top of a rock. Let x and θ be as pictured.



(a) Find a function $L(x)$ which computes Lee's viewing angle θ as a function of the labeled distance x. What is the angle θ if $x = 2$ ft.,10 ft., 20 ft.?

(b) Assume Lee begins standing at the location $x = 20$ at time $t = 0$. He is getting nervous and starts moving backward and forward so that his location $x = x(t)$ in front of the rock after t seconds is given by $x = x(t) = 20 + 18\sin(\frac{t}{2})$.

i. Draw a little picture indicating what Lee is doing. Also sketch a NICE ACCURATE picture of the graph of $x(t)$ and LABEL the axes.

ii. What is the closest and farthest Lee will be from the rock?

iii. Where is Lee located at times $t = 2, 8$ seconds?

iv. What is Lee's viewing angle at times $t = 2, 8$ seconds?

v. How long does it take Lee to complete one cycle of his motion; i.e. what is the period for Lee's motion?

vi. During the first 20 seconds, when is Lee 4 feet from the rock?

(c) Assume the situation in b. Define a NEW FUNCTION by the formula $A(t) = L(x(t))$.

i. Write down a formula for this function. Explain in words what this function tells you?

ii. Use a graphing device to sketch a NICE ACCURATE graph of $A(t)$ on the domain $0 \leq t \leq 8\pi$; use a "y-range" of $[0, 0.6]$. LABEL the axes. Is this a sinusoidal function? Is this a periodic function? Is this a weird function? Justify your answers.

iii. Focus on the times $2\pi \leq t \leq 4\pi$. How do the parts of the graph of $A(t)$ which are *increasing* and *decreasing* correlate with Lee's motion *toward* and *away* from the rock?

iv. During the first minute (i.e. 60 seconds), what is the range of possible viewing angles for Lee?

v. During the first 15 seconds, use your graphing device to determine when Lee's viewing angle is 23 degrees. Determine Lee's location at these times.

19.* Suppose

$$T(t) = 23\sin\left(\frac{2\pi}{24}(t-7)\right) + 66$$

is the temperature (in degrees Fahrenheit) at time t, where t is measured in hours after midnight on Sunday. You paint the exterior door to your house at 5pm on Monday. The paint information states that 48 hours of 75 degree drying time is required; i.e. you can only count time periods when the temperature is at least 75 degrees. When will the door be dry?

20. Return to Exercise 3.5.7. Answer these additional questions:

(a) During the first 2 minutes of the test, when will the volume per breath be at least 1 liter?

(b) During the first 2 minutes of the test, when will the volume per breath be no more than 1 liter?

21. Some population models (especially in predator–prey situations) lead to populations which vary sinusoidally. Suppose you begin observing the population of sharks in an area and make the following estimates: The maximum population was 14,000 and it occured at $t = 6$ months. Fourteen months later (at $t = 20$ months), the population bottomed out at a minimum of 6,000. Assume that the population $P(t)$ varies sinusoidally with time.

    (a) Sketch a graph of P versus t over the interval $0 \leq t \leq 36$ months. Label the coordinates at each maximum and minimum of $P(t)$.

    (b) Find a formula for $P(t)$.

    (c) When is the first time after $t = 6$ that the population was 12,000?

22. For several hundred years astronomers have kept track of the number of solar flares, or "sunspots", which occur on the surface of the sun. The number of sunspots counted in a given year varies sinusoidally from a minimum of about 10 per year to a maximum of about 110 per year. Between the maximums that occured in the years 1750 and 1948, there were 18 complete cycles.

    (a) What is the period of the sunspot cycle?

    (b) Assume a sinusoidal function $s(x)$ computes the number of sunspots in a given year x. Sketch a graph for two sunspot cycles, starting from 1948.

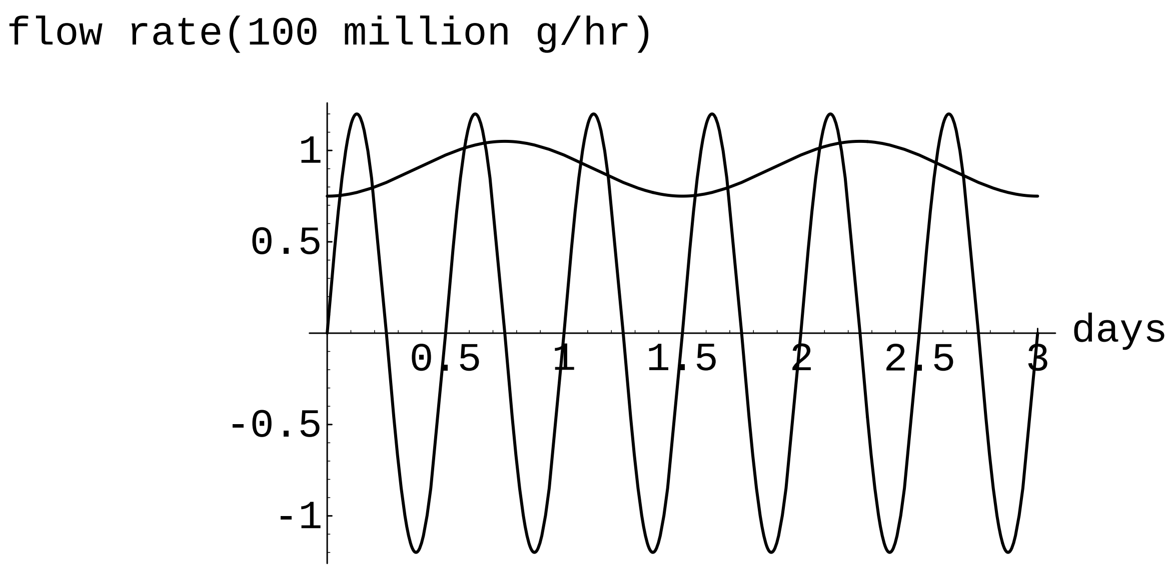
    (c) Write an equation expressing the number of sunspots per year in terms of the year.

    (d) How many sunspots would you expect in the year 2000?

    (e) What is the first year after the year 2000 in which the number of sunspots will be about 35?

23.* [1] You have been hired to aid in a study of the salinity of a bay near the mouth of a tidal river. A tidal river is one that empties into the ocean or an oceanic bay and is so influenced by the tides that its flow is noticeably affected (we have 'um in Washington). That is, the change in water level in the body of water into which the river flows affects the flow of water within the river.

[1] Adapted from *Calculus, An active approach with projects*, Ithaca College Calculus Group, Wiley, 1994



You have collected two kinds of data for the three-day period. The first is a measure of the rate at which the fresh water flows from the river. This can be interpreted as the part of the river's flow rate that is not due to the effect of the tides. The second is a measure of the effect of the tides. That is, if the river were totally stagnent (not flowing at all), then water would flow into and out of its mouth solely as a result of the tidal rise and fall of the water level in the bay. Both data are given in the graphs.

(a) Which curve is the flow rate $r(t)$ of fresh water from river? Which curve is the rate $o(t)$ of tide water from the river?

(b) During the first 6 hours, is tide water flowing in or out of the river?

(c) Find formulas for $r(t)$ and $o(t)$.

(d) Use a graphing device to plot the net rate of flow of water from the river into the bay over the three-day period. What is the maximum and minimum net rate of flow?

(e) Is the function in b. a sinusoidal function? Is it a periodic function?

(f) During the first day, determine when water is neither flowing in nor out of the river. (You will need to use your graphing device.)

24. In order to conserve water, Mary has installed a large "rainbarrel" to hold rainwater for her garden plants. This is in the shape of a cylinder and the water level at the start of April is 80 cm. During the month of April, assume the water level in the barrel is effected by only two things: (i) on rain days the water level rises and she does not water her plants; (ii) on

clear days she waters her plants and the water level in the barrel is falling. We will ignore evaporation. The function

$$r(t) = \frac{2}{3}\sin(\frac{\pi}{17}t) + \frac{1}{4} \quad cm/day$$

gives the rate of change of the water level on the t^{th} day. A positive (resp. negative) rate of change means the water level is rising (resp. falling).

(a) Sketch the graph of $r(t)$ during April.

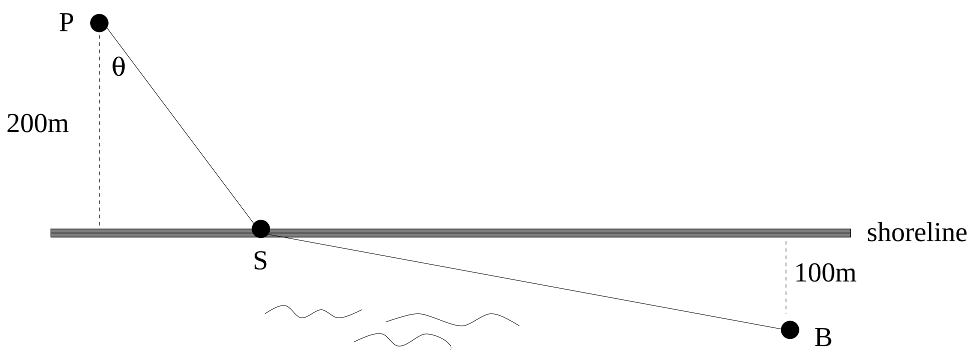
(b) Find the largest rate at which the water level is rising and falling in the rainbarrel during April.

(c) During how many days of April is it raining?

(d) During April, when will the water level reach its highest point?

(e) * At the end of April, is the water level above or below its level on April 1?

25. Suppose that you want to run a cable from a power source P out to an offshore buoy B. The power source is located on land, 200 meters from a straight section of the shoreline, and the buoy is located in the water, 100 meters out from a point 600 meters downshore from P; see picture. The cable will run in a straight segment from P to the point S on the shoreline, and then from S in another straight segment out to B.



(a) Find a formula for the total length ℓ of the cable as a function of the pictured angle θ.

(b) What value of θ will minimize the total length? Note: You do not need to use a graphing calculator (or calculus) to minimize the length. Use common sense instead. (Hint: What is the shortest distance between two points?)

26.* According to Exercise 12, §3.4 we have the identity: $\cos(t) = \sin(\frac{\pi}{2} + t)$. If you combine this with (3.4.7), we have this equality for ALL values of t:

$$\sin(\frac{\pi}{2} + t) = \sin(\frac{\pi}{2} - t).$$

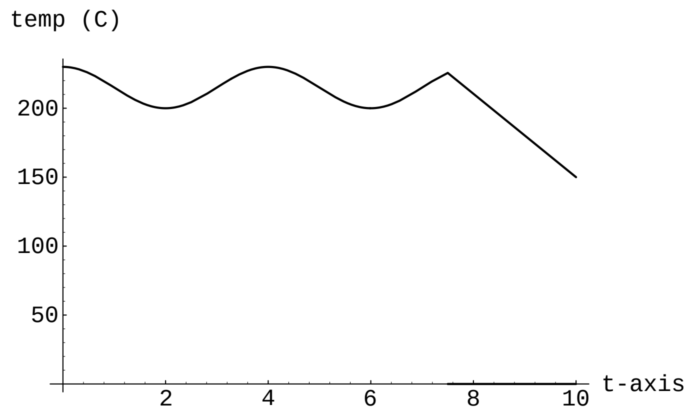
Explain what is WRONG with the following reasoning: Apply "$\sin^{-1}$" to each side, obtaining the equation

$$\frac{\pi}{2} + t = \frac{\pi}{2} - t,$$

for all t values. Subtract $\frac{\pi}{2}$ from each side, set $t = 1$ and conclude that 1=-1.

27. You are preparing to bake a pizza at 230^oC. You hear a loud "POP" and notice the digital readout for the oven temperature has started to display changing temperatures. You record these temperatures over a 10 minute period. The resulting data is modeled by a function $P(t)$ computing the oven temperature (^{o}C) at time t minutes after the "POP". Assume these five facts:

- Here is the graph of the function $P(t)$ on the domain $0 \leq t \leq 10$:



.

- The function $P(t)$ is sinusoidal during the time $0 \leq t \leq 7.5$ minutes.
- The function $P(t)$ is linear function during the times $7.5 \leq t \leq 10$ minutes.
- The maximum oven temperature is $230^o C$ and first occurs at time $t = 0$. The minimum oven temperature is $200^o C$ and first occurs at time $t = 2$.
- The temperature at time $t = 10$ is $150^o C$.

(a) On the domain $0 \leq t \leq 7.5$, we know that the function has the form

$$P(t) = A \sin(\frac{2\pi}{B}(t - C)) + D,$$

for some constants A, B, C, D. Find these four constants.

(b) What is the oven temperature at time $t = 7.5$ minutes?

(c) What is the multipart rule for the function $P(t)$ on the domain $0 \leq t \leq 10$?

(d) During the first 10 minutes, find the total amount of timewhen the oven temperature is below 220^oC.

(e) The conversion from C to F degrees is given by the formula: $F = \frac{9}{5}(C - 100) + 212$. Write down the multipart rule for a function $F(t)$ that computes the temperture of the oven in ^{o}F as a function of time t in minutes, on the domain $0 \leq t \leq 10$.

28. The angle of elevation of the sun above the horizon at noon is approximately a sinusoidal function of the day of the year. The amplitude of this function is 23.452^o; this is the angle at which the earth's axis is tipped. The period is 365.25 days. The maximum elevation occurs on June 21, the 172nd day of the year. If the latitude where the observation is made is θ degrees, then the mean elevation of the sun is $90 - \theta$ degrees.

(a) Find a sinusoidal function $E(t)$ which approximates the elevation of the sun on day t of the year. (Your answer will involve the unknown latitude θ.)

(b) At noon on the 74th day of the year, you measure the elevation of the sun to be $39^o 39$' on the UW campus. Use your answer to part (a) to determine the latitude θ of the UW campus. Give your answer in degrees and minutes.

(c) If the latitude $\theta = 40^o$, find the time(s) when the angle of elevation of the sun at noon is 35^o.

(d) Imagine you are way up north in Alaska with a latitude $\theta = 70^o$. Sketch the graph of the function $E(t)$. Find the day(s) when the sun never rises.