2.8 Inverse Functions

1. Show that the functions $y = x^3$, $y = 2x + \pi$ and $y = -2x$ are one-to-one by checking their graphs.

2. A function is called *onto* if every horizontal line will intersect the graph at least once.

   (a) Show that x^3 is onto and that x^2 is not onto.

   (b) Draw pictures of a graph (you do not need an algebraic formula - just the graph) which is (i) one-to-one and onto (don't use x^3) (ii) one-to-one but not onto (iii) not one-to-one but onto (iv) neither one-to-one nor onto (don't use x^2)

3. Sometimes there is a way of getting an algebraic formula for $f^{-1}(y)$ given a formula for the one-to-one function $f(x)$. The idea is to solve the equation $y = f(x)$ for x in terms of y, and that's $f^{-1}(y)$. Here's an example: If $f(x) = 4x - 3$, then to find $f^{-1}(y)$, we rewrite $y = 4x - 3$ as $\frac{y+3}{4} = x$.

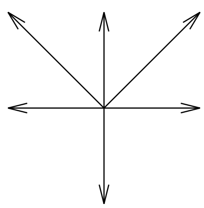
   (a) Find $f^{-1}(y)$ if $f(x) = 2x + 3$.

   (b) Find $f^{-1}(y)$ if $f(x) = (x + 2)^3$.

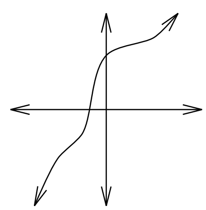
   (c) Find $f^{-1}(y)$ if $f(x) = \frac{1}{x}$. (You get a surprising answer - describe that graphically!)

4. Which of the following graphs are one-to-one. If they are not one-to-one, section the graph up into parts that are one-to-one.

A



B



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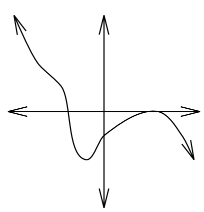

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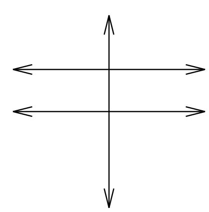
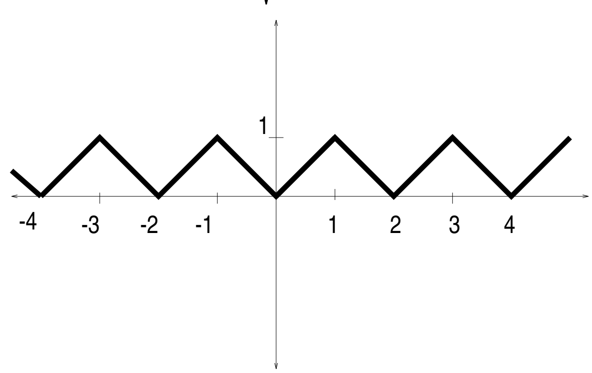


5. Consider the *sawtooth function* pictured.



   (a) Is this function one-to-one on the domain of all real numbers?

(b) Find the inverse function on the domain $0 \leq x \leq 1$ algebraically and graphically. (In the process, you will need to explain why the function is one-to-one, compute its range and write out a formula for the inverse function, including information about it's domain.)

(c) Find the inverse function on the domain $-1 \leq x \leq 0$ algebraically and graphically. (In the process, you will need to explain why the function is one-to-one, compute its range and write out a formula for the inverse function, including information about it's domain.)

(d) Find the inverse function on the domain $2 \leq x \leq 3$ algebraically and graphically. (In the process, you will need to explain why the function is one-to-one, compute its range and write out a formula for the inverse function, including information about it's domain.)

6. For each of the following functions: (1) sketch the function, (2) find the inverse function(s), and (3) sketch the inverse function(s). In each case, indicate the correct domains and ranges. Finally, make sure you test each of the functions you propose as an inverse with the following compositions:

$$f(f^{-1}(x)) \stackrel{?}{=} x$$

and

$$f^{-1}(f(x)) \stackrel{?}{=} x.$$

(a) $f(x) = 3x - 2$
(b) $f(x) = -7x + 3$
(c) $f(x) = \frac{1}{3x-2}$
(d) $f(x) = \frac{1}{x+3}$
(e) $f(x) = \frac{3x+2}{2x-5}$
(f) $f(x) = x^2 + 5$
(g) $f(x) = -x^2 + 3$
(h) $f(x) = 2 + 5x - 3x^2$
(i) $f(x) = 5x^2 + 10x + 2$
(j) $f(x) = \sqrt{3 - x}$
(k) $f(x) = \sqrt{4 - x^2}$, where $0 \leq x \leq 2$
(l) $f(x) = x^{\frac{1}{3}} + 1$

7. For each positive integer $n = 1, 2, 3, \ldots$, define a function

$$y = H_n(p) = \frac{p^n}{26^n + p^n}.$$

(a) Use a graphing device to sketch the graphs of $H_1(p), H_2(p), H_3(p)$ in the same coordinate system on the domain $0 \leq p \leq 100$.

(b) Find the range of each of the functions in (a).

(c) Find formulas for the inverse functions H_n^{-1}, for $n = 1, 2, 3$. What are the domains and ranges of these inverse functions.

(d) Verify that

$$H_n^{-1}(H_n(p)) = p$$

and

$$H_n(H_n^{-1}(y)) = y,$$

for $n = 1, 2, 3$.

(e) Use a graphing device to sketch the graphs of $H_n^{-1}(y)$, for $n = 1, 2, 3$ on each of their domains.

(Note: The functions $H_n(p)$ arise when studying the oxygen binding ability of hemoglobin in human red blood cells. We will return to this in the problems for §5.1.)

8. Let $f(x) = \frac{2}{3x-4}$ on the largest domain for which the formula makes sense.

   (a) Find the domain and range of $f(x)$, then sketch the graph.

   (b) Find the domain, range and rule for the inverse function f^{-1}, then sketch its graph.

9. A biochemical experiment involves combining together two protein extracts. Suppose a function $\phi(t)$ monitors the amount (nanograms) of extract A remaining at time t (nanoseconds). Assume you know these facts:

   - The function ϕ is invertible; i.e. it has an inverse function.
   - $\phi(1) = 5, \phi(2) = 3, \phi(3) = 1, \phi(4) = 0.5, \phi(10) = 0.$

   (a) At what time do you know there will be 3 nanograms of extract A remaining?

   (b) What is $\phi^{-1}(0.5)$ and what does tell you?

   (c) (True or False) There is exactly one time when the amount of extract A remaining is 4 nanograms.

   (d) Calculate $\phi(\phi^{-1}(1)) =$

   (e) Calculate $\phi^{-1}(\phi(6)) =$

   (f) What is the domain and range of ϕ?

10. For this problem, let $y = f(x) = 2 - 2x - \frac{1}{3}x^2$.

    (a) Sketch an accurate graph of $y = f(x)$ on the domain $-8 \le x \le 5$. Explain why the function f is NOT invertible on this domain.

    (b) Divide the domain $-8 \le x \le 5$ into two smaller domains: $-8 \le x \le -3$ and $-3 \le x \le 5$. Explain why the rule $f(x)$ on each of these two smaller domains defines a function which has an inverse.

    (c) For each of the two functions in (b) (obtained by restricting the rule $f(x)$ to one of the two smaller domains), find the formula $x = f^{-1}(y)$ for the inverse function. Graph the two resulting inverse functions, specifying the domain and range in both cases.

11. Clovis is standing at the edge of a cliff, which slopes 4 feet downward from him for every 1 horizontal foot. He launches a small model rocket from where he is standing. Also, with the origin of the coordinate system located where he is standing, the path of the rocket is described by the formula $y = -2x^2 + 120x$.

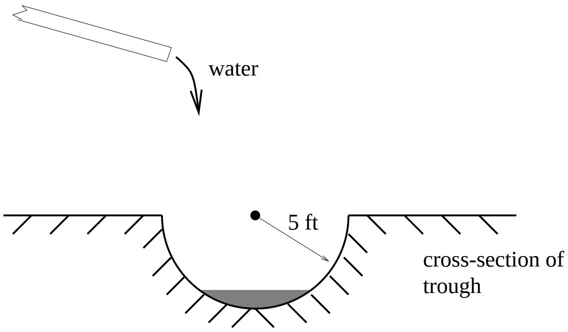
    (a) Give a function $h = f(x)$ relating the height h of the rocket above the sloping ground to its x-coordinate.

    (b) Find the maximum height of the rocket above the sloping ground. What is its x-coordinate when it is at its maximum height?

    (c) While the rocket is still going up, Clovis measures its height h above the sloping ground. Give a function $x = g(h)$ relating the x-coordinate of the rocket to h.

    (d) Does this function still work when the rocket is going down? Explain.

12. A trough has a semicircular cross section with a radius of 5 feet. Water starts flowing into the trough in such a way that the depth of the water is increasing at a rate of 2 inches per hour.



(a) Give a function $w = f(t)$ relating the width w of the surface of the water to the time t, in hours. Make sure to specify the domain and compute the range too.

(b) After how many hours will the surface of the water have width of 6 feet?

(c) Give a function $t = f^{-1}(w)$ relating the time to the width of the surface of the water. Make sure to specify the domain and compute the range too.