2.7 Polynomial Modeling

1. Return to the list of twelve polynomials given at the start of §2.7.1. What is the degree of each polynomial?

2. An open box is made out of rectangular material 20×30 inches by cutting squares from each corner.

   (a) Find a polynomial $V(x)$ in the variable x, for the volume of the box, if x represents the side length of a cutout square.

   (b) If the cutouts are 2 inches on each side, what is the volume of the box?

   (c) Find the largest domain of x values for the function $V(x)$ which makes physical sense.

   (d) If the volume is 1000 cubic inches, what are the feasible dimensions of the box.

3. A right triangle has hypotenuse (long side opposite the right angle) 4 feet longer than one of the sides. Assume the area of the triangle is 24 sq. ft.

   (a) Sketch a picture of the situation, labeling the dimensions.

   (b) Use the formula for the area of a triangle ($\frac{1}{2}(base)(height)$) and the given constraints (the area and the side length information) to obtain an equation in a single variable. This equation will involve a square root symbol, which can be eliminated by squaring both sides of the equation.

   (c) Fully factor the resulting polynomial equation in the previous step.

   (d) Find the dimensions of the triangle.

4. Recall, we reviewed the fact that two distinct points in the plane determine a unique line. Suppose we specify the points (1,0), (2,0) and (3,0) on the x-axis.

   (a) How many lines can you draw through these three points?

   (b) Can you find a second degree polynomial with these three roots?

   (c) Can you find a third degree polynomial with these three roots? Is this polynomial unique?

   (d) Can you find a fourth degree polynomial with these three roots? Is this polynomial unique?

5. Fully factor each of these polynomials, find all roots and graphically indicate where the graphs are above and below the x-axis:

   (a) $y = x^3 - 2x + 4x^2 - 8$.

   (b) $y = x^3 + 2x^2 - 6x - 7$.

   (c) $y = x^3 + 2x^2 - 6x + 7$.

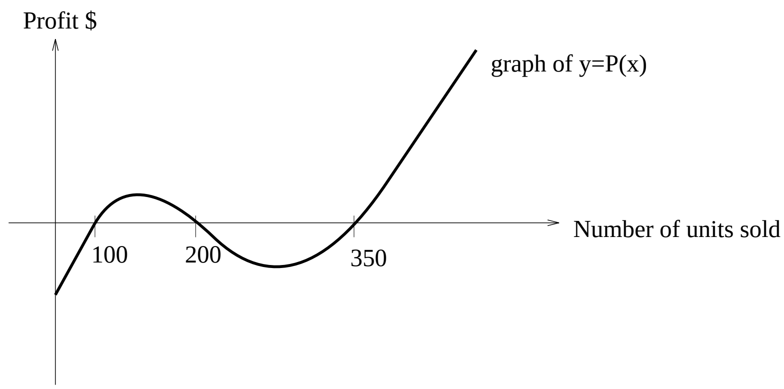
   (d) $y = x^2 + \pi^3$.

   e. $y = x^2 + 2x - 11$.

   f. $y = x^6 + x - 1$.

   g. $y = x^5 - 4x^4 + 3x^3 - x^2 + 2 + 1$.

6. The function $y = P(x)$ gives the profit (in dollars) on the sale of x hamburgers at the *Burger Barn*.



(a) (True or False) The function $y = P(x)$ is a quadratic function; give a reason.

(b) The "break even points" are the roots of the function $y = P(x)$; i.e. the solutions to the equation $0 = P(x)$. What are the break even points for this example?

(c) Suppose that you know two facts: (1) The function $y = P(x)$ is a third degree polynomial; (2) The profit on the sale of 0 hamburgers is -\$2000 (i.e. $P(0) = -2000$). Explicitly determine $y = P(x)$. What is the profit on the sale of 500 hamburgers?

7. Short answer.

(a) Describe all possible quadratic polynomials having 1 and -1 as roots.

(b) Describe all possible quadratic polynomials having 1 and -1 as roots and a graph opening upward.

(c) Describe all possible quadratic polynomials having 1 and -1 as roots and a graph opening downward.

(d) Describe all possible quadratic polynomials having 1 and -1 as roots and a graph crossing the y-axis at 7.

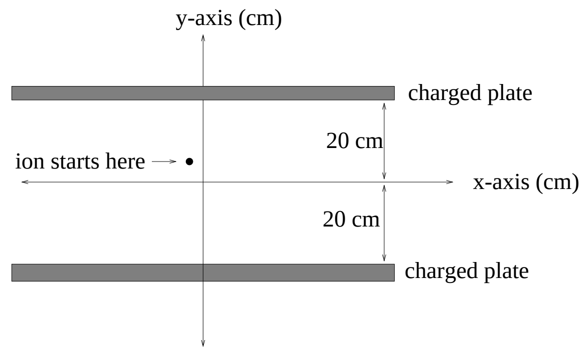
(e) Suppose $f(x) = x^2$ and $g(x)$ is a quadratic function whose graph opens upward. Will these graphs of these two functions intersect? Try to give a graphical argument and an argument using equations.

8. *Billysoft*, a new startup software company, is selling shares of stock to the public. As a cautious investor, you decide to purchase one-share of the stock and track it's price for 200 days. Let $p(x)$=(closing price on day x)-(price you paid). So, $p(1)$ is the gain you make owning the stock one day and $p(200)$ is the gain you make owning the stock 200 days. Assume these three facts: (a) The function $p(x)$ is a polynomial function of degree four; (b) The only x values for which the stock has the same value as the original purchase price are $x = 0, 110, 160$ and 200; (c) Fifty days after you purchased your stock, it is worth \$99.50 and you originally paid \$50.

(a) Find a formula for $p(x)$. How much money have you made 10 days after your purchase?

(b) During the first 200 days of your stock ownership, how much of the time are you losing money on your investment?

9. This problem will require a graphing device to get started. A positively charged ion is located between two metal plates which are 40 cm apart. A varying $\pm$ charge is applied to each plate, causing the ion to follow a curved path.



If coordinates are imposed as pictured (with units of cm on each axis), the path of the ion is given by the graph of the function

$$\begin{aligned} p(x) &= 3.60095x - 1.21401x^2 - 2.90611x^3 + 1.46604x^4 + \\ &\quad +0.260556x^5 - 0.249583x^6 + 0.0446032x^7 - 0.0024504x^8. \end{aligned}$$

The ion is initially located at the position $(-2, p(-2))$ and once the ion strikes a plate it will stop moving. Assume the x-coordinate of the ion t seconds after applying voltage to the plates is given by $x = x(t) = -2 + 20t$.

(a) Determine where and when the ion is equidistant from each plate?

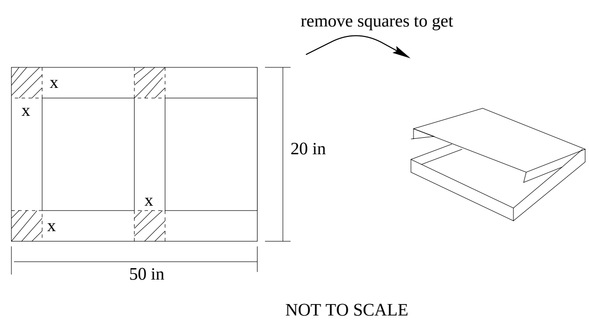
(b) Find a formula for the y-coordinate of the ion at time t.

(c) Determine when and where the ion strikes a charged plate.

(d) During the time the ion is is motion between the two plates, how much of the time is it closer to the bottom plate than to the top plate?

(e) Where and when is the ion closest to the top plate?

10. *Pagliacci Pizza* has designed a cardboard delivery box from a single piece of cardboard, as pictured.



(a) Find a function $v(x)$ that computes the volume of the box in terms of x.

(b) If you want the box to enclose 500 cu. inches, what size pie can be delivered in the box?

(c) If you want the box to enclose the largest total volume, what size pie can be delivered in the box?

11.* This exercise is lengthy and introduces the idea of approximating a given graph by polynomial graphs. A graphing device is required. The guiding question is this: How can you compute $\sqrt{3}$?

(a) Punch "$\sqrt{3}$" into your hand-held calculator and write down the answer.

(b) Sketch a very accurate graph of $y = f(x) = \sqrt{x}$ on the domain $0 \leq x \leq 10$ and on the domain $2.8 \leq x \leq 3.2$.

(c) Sketch accurate graphs of each of these five polynomials on the domain $0 \leq x \leq 10$ and on the domain $2.8 \leq x \leq 3.2$:

$$\mathcal{T}_0 f(x) = 2$$

$$
\begin{aligned}
\mathcal{T}_1 f(x) &= 2+\frac{1}{4}(x-4)\\
\mathcal{T}_2 f(x) &= 2+\frac{1}{4}(x-4)+(\frac{-1}{64})(x-4)^2\\
\mathcal{T}_3 f(x) &= 2+\frac{1}{4}(x-4)+(\frac{-1}{64})(x-4)^2+(\frac{1}{512})(x-4)^3\\
\mathcal{T}_4 f(x) &= 2+\frac{1}{4}(x-4)+(\frac{-1}{64})(x-4)^2+(\frac{1}{512})(x-4)^3+(\frac{-5}{16384})(x-4)^4
\end{aligned}
$$

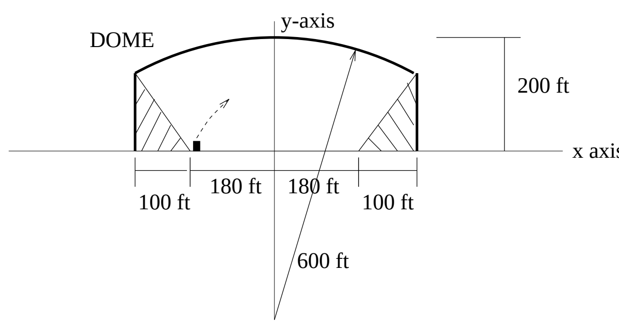
(d) Compute the value of each of these polynomials and $f(x)$ when $x = 4$. Explain why this tells us that each of the five polynomial graphs and the graph of $y = f(x)$ all pass through the point $P = (4, 2)$.

(e) In a common coordinate system, compare your graphs of these polynomials and the graph of $f(x)$ above 3 on the x-axis; what is happening?

(f) Evaluate each of the five polynomials at $x = 3$ and compare to $f(3)$. What is your conclusion?

The polynomials introduced in this problem are called the *Taylor polynomials* for $y = \sqrt{x}$.

12. The basic shape of a Sportsdome is a cylinder with a spherical cap. A cross-section is given below. Assume the dimensions given and the imposed coordinate system.



(a) Edgar steps to the plate and hits a ball which follows the graph of the function

$$y = -\frac{x^2}{121} - \frac{49x}{242} + \frac{28353}{121}.$$

Show the ball hits the top of the dome.

(b) The ball is 4 feet high when Edgar hits it; where is Edgar standing?

(c) Assume backspin on the ball causes it to drop straight down to the field from the point it hits the roof. Where does it land on the field?

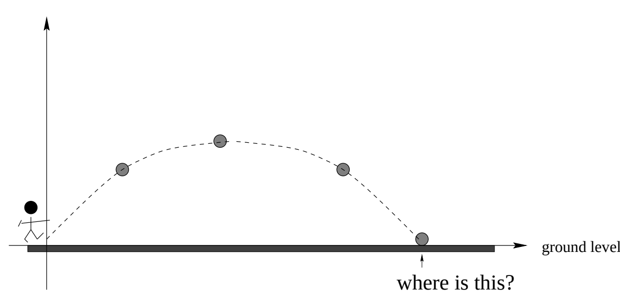
(d) If the dome had no roof, where would Edgar's ball land?

13.* Start with the function $y = \frac{1}{1-x}$ and specify a positive integer n. Carry out long division (with remainder) to write

$y = \frac{1}{1-x} = p_n(x) + \text{(remainder term)}$

for some polynomial $p_n(x)$ of degree n. Use your graphing device to simultaneously plot the graphs of $y = \frac{1}{1-x}$ and $p_n(x)$ for $n = 1, 2, 3, 4$. Explain what is happening.

14. A ball is kicked and follows a parabolic path given by the graph of function $y = f(x) = -0.03557x^2 + x$, where x represents the horizontal location of the ball in front of the kicker (in units of feet).



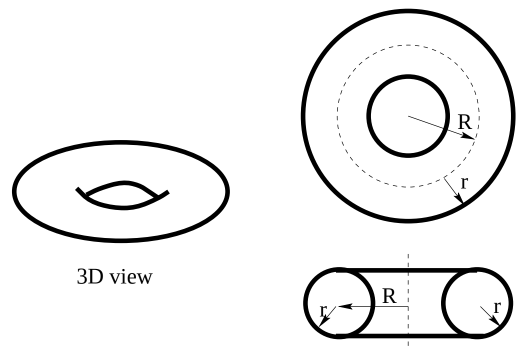
(a) Where does the ball hit the ground?

(b) Use a graphing device to approximate the solutions to $-0.03557x^2 + x - 5 = 0$. What is the physical meaning of your solutions?

(c) Assume that the location of the tossed ball at time t seconds is given by $P(t) = (21.21t, -16t^2 + 21.21t)$. Use a graphing device to approximate WHEN and WHERE the ball is located the instant it is 20 feet (along a straight line) from the kicker's foot.

15.* The Washington State Ferry system has hired you to design a new life ring. You can assume these facts:

- This ring is constructed from styrofoam which has a density of 0.1 g/cm^3.
- The ring has the shape of a *torus*, which looks like a big "donut" (as pictured). The dimension R (resp. r) is called the core radius (resp. cross-sectional radius). The volume of the torus is $V = 2\pi^2 R r^2$.
- The hole in the ring is 24 inches across.
- The mass of the water displaced by a fully submerged life ring must exceed the mass of the life ring by at least 200 pounds.



What are the minimum dimensions of the life ring? (You will ultimately obtain a third degree polynomial and need to find a root; use a graphing device. Be careful about the units. It is a good idea to either convert everything to units of inches and pounds, or convert everything to the metric system.)

16. (a) Use a graphing device to approximate the negative root of $y = f(x) = x^5 + x + 1$.

(b) The exact value for the root is

$$c = \frac{1}{3} + \frac{\sqrt[3]{2}}{3\sqrt[3]{-25 + 3\sqrt{69}}} + \frac{\sqrt[3]{-25 + 3\sqrt{69}}}{3\sqrt[3]{2}}.$$

As an exercise, try this on your calculator and compare your answers.

(c) How many (real) roots do you think $f(x)$ will have; try to give a graphical reason for your answer.