2.5 Three Construction Tools

1. Begin with the graph of the equation $y = x$. Let m, h and k be fixed constants.

   (a) Using the ideas of this section, how can we relate the graphs of $y = x$ and $y = mx$?

   (b) Using the ideas of this section, how can we relate the graphs of $y = x$ and $y = m(x - h) + k$?

   (c) Assuming $m \neq 0$ and $b \neq 0$, how can we relate the graphs of $y = mx + b$ and $y = \frac{1}{m}x - \frac{b}{m}$? Do the two graphs intersect? If so, find where they intersect. (Hint: You can relate the two graphs by reflection across a line.)

2. On a single set of axes, sketch a picture of the graphs of the following four equations: $y = -x + \sqrt{2}$, $y = -x - \sqrt{2}$, $y = x + \sqrt{2}$ and $y = x - \sqrt{2}$. These equations determine lines, which in turn bound a diamond shaped region in the plane.

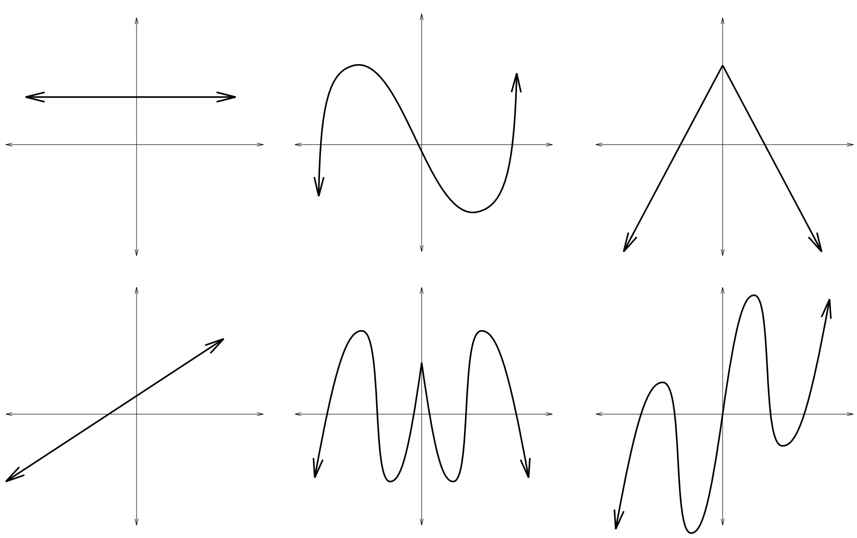
   (a) Show that the unit circle sits inside this diamond tangentially; i.e. show that the unit circle intersects each of the four lines exactly once.

   (b) Find the intersection points between the unit circle and each of the four lines.

   (c) Construct a diamond shaped region in which the circle of radius 1 centered at (-2,-1) sits tangentially. Use the techniques of this section to help.

3. Return to Exercise 2.2.10. Carry out the operations of Tables 2.5.1-2.5.3, with $c = 2$, for this example.

4. If a function $f(x)$ has the property that $f(x) = f(-x)$ (resp. $f(-x) = -f(x)$) for all x in the domain, we call f an *even function* (resp. *odd function*). What does even (resp. odd) mean in terms of graph symmetry? Which of the following pictured curves are graphs of even or odd functions?



5. Consider the function $y = f(x)$ with multipart definition

$$f(x) = \begin{cases} 0 & \text{if } x \leq -1 \\ 2x + 2 & \text{if } -1 \leq x \leq 0 \\ -x + 2 & \text{if } 0 \leq x \leq 2 \\ 0 & \text{if } x \geq 2 \end{cases}$$

(a) Sketch the graph of $y = f(x)$.

(b) Is $y = f(x)$ an even function; the definition of an even function is given in the previous problem.

(c) Sketch the reflection of the graph across the x-axis and y-axis. Obtain the resulting equations for these reflected curves.

(d) Sketch the vertical dilations $y = 2f(x)$ and $y = \frac{1}{2}f(x)$.

(e) Sketch the horizontal dilations $y = f(2x)$ and $y = f(\frac{1}{2}x)$.

(f) Find a number $c > 0$ so that the highest point on the graph of the vertical dilation $y = cf(x)$ has y-coordinate 11.

(g) Using horizontal dilation, find a number $c > 0$ so that the function values $f(\frac{x}{c})$ are non-zero for all $-\frac{5}{2} < x < 5$; sketch a picture.

(h) Using horizontal dilation, find a number $c > 0$ so that the function values $f(\frac{x}{c})$ are non-zero for all $-\frac{1}{6} < x < \frac{1}{3}$; sketch a picture.

6. Let a, b be positive constants. We will work with the *ellipse* equation

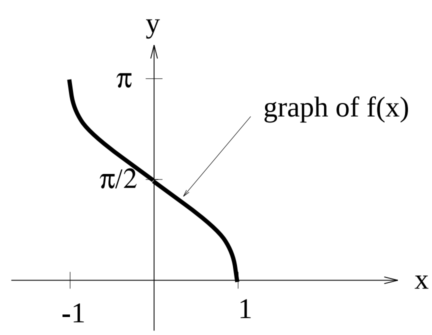
$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1.$$

(a) Solve the equation for y in terms of x; you should get two different functions.

(b) Use a graphing device to study the graphs of the functions when a, b are either 1/2 or 2; there are several cases to consider. What is the connection between these graphs and the set of solutions of the original equation?

(c) Describe how to use horizontal and vertical dilation to obtain the pictures in the previous part, starting from the unit circle (the circle of radius 1 centered at the origin).

7. The graph of a function $y = f(x)$ is pictured. It's domain is the interval $-1 \leq x \leq 1$. Sketch the graph of $y = \frac{1}{\pi}f(3x) - 0.5$. Find the largest possible domain of the function $y = g(x) = \sqrt{\frac{1}{\pi}f(3x) - 0.5}$.



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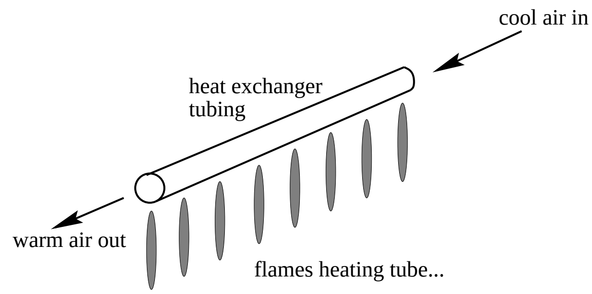
(a) Go to Table 2.5.1 and add a new column called "picture formula" where the multipart formula for the function in the "picture column" is given.

(b) Go to Table 2.5.2 and add a new column called "picture formula" where the multipart formula for the function in the "picture column" is given.

(c) Go to Table 2.5.3 and add a new column called "picture formula" where the multipart formula for the function in the "picture column" is given.

9. A typical home gas furnace contains a *heat exchanger* which consists of a bunch of cylindrical metal tubes through which air passes as the flames heat the tube. As a furnace ages, these tubes can deform and crack, allowing deadly carbon monoxide gas to leak into the house. Suppose the cross-section of a heat exchanger tube is a circle of radius 5 cm when



it is brand new. As the furnace ages, assume the bottom semicircular cross-section deforms according to the vertical dilation principle 2.5.4 with a deformation constant of

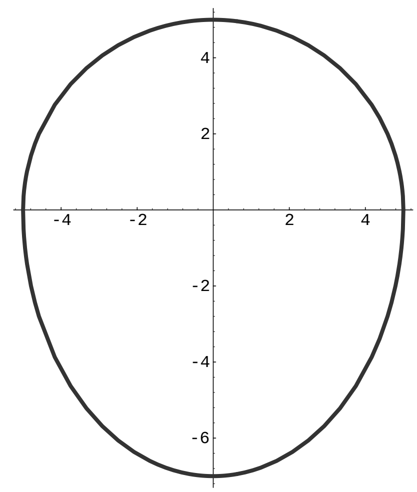
$$c = c(t) = 1 + \frac{1}{500}(t^2 + t)$$

after t years. The top semicircular cross-section does not deform.

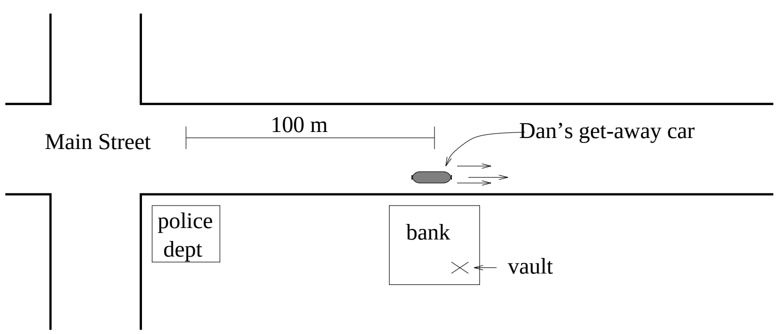
(a) Sketch an accurate picture of the cross-section of the heat exchanger tubing after 10 and 15 years.

(b) Suppose the metal in the tubing will crack if it deforms more than 3 cm out of the original shape. When will the exchanger crack?

(c) When will the cross-section of the heat exchanger tubing look like the picture at right?



10. Dastardly Dan has just robbed a bank. Unfortunately, Dan's getaway car doesn't have much pickup. With the accelerator floored, the distance he has traveled from the bank down Main Street is given by $y = t^2$, where y is the distance measured in feet and t is the time in seconds since he started the car.



This problem will place the graphical operations of dilation and shifting into a context and requires a graphing device. You should start with a viewing window of $-40 \leq t \leq 70$ by $-10 \leq y \leq 1000$.

(a) Sketch a graph of the distance Dan has traveled down Main Street, as a function of time. What are the coordinates of lowest point on the graph?

(b) Suppose we want to graph the distance Dan has traveled in his car from the time he *leaves the vault*, rather than from the time he started the car. It takes him 15 seconds to get from the bank vault to his car.

i. How far has Dan traveled down Main Street 15 seconds after he left the vault?

ii. 20 seconds after leaving the vault? 30 seconds after leaving the vault? t seconds after he left the vault?

iii. This last expression is the equation of the distance Dan has traveled down Main Street t seconds after he *leaves the vault*. Sketch a graph of this function in the same coordinate system you used in a. How do these two graphs differ? What is the lowest point of this new graph?

(c) Suppose Dan is being pursued by Carla the cop. Carla is Canadian, and so she wants to know how far Dan has traveled in *meters* rather than feet.

i. Fifteen seconds after Dan left the vault, how many meters has he traveled down Main Street?

ii. After 20 seconds? After 30 seconds? After t seconds?

iii. This last equation gives the distance Dan has traveled down Main Street in meters since he left the vault. Graph this equation using your graphing device (together with the last two graphs) in the same coordinate system and label each curve. How does this graph differ from the previous two graphs? What is the lowest point of this new graph?

(d) The police station is 100 meters behind the bank. Carla wants to know how far Dan is from the police station.

i. How far (in meters) is Dan from the police station 15 seconds after he leaves the vault?

ii. After 20 seconds? After 30 seconds? After t seconds?

iii. This last equation gives the distance (in meters) from Dan to the police station t seconds after he has left the vault. Graph this with your other graphs. Label this last graph and discuss what has changed? What is the lowest point of this new graph?

(e) Now Carla the cop is giving chase. She leaves from in front of the police station 30 seconds after Dan leaves the bank's vault. She drives down Main Street at a constant 45 miles per hour (Carla may be Canadian, but she's driving an American car). Find a formula for the distance (in meters) Carla has traveled t seconds since Dan left the vault.

(f) Graph the distances Carla and Dan have traveled down Main Street against the time since Dan has left the vault. That is, graph the line and the parabola from e. and d., for the specified t values. Label both graphs with their equations.

(g) Does Carla catch Dan? Explain, using your graph, how you know.

(h) We want to find out how close Carla gets to Dan. To do this, we'll find how far Dan is in front of Carla.

i. How far is Dan in front of Carla 30 seconds after he leaves the vault?

ii. After 40 seconds? After 50 seconds? How about t seconds after he left the vault?

iii. This last expression is the equation of the distance Dan is in front of Carla. Graph this function and label it.

iv. Find the *closest* Carla comes to Dan and when this happens. Do this by finding the lowest point of the graph you just found.

(i) Terry the State Trooper is parked on the side of the road and is passed by a speeding Dan 45 seconds after Dan left the vault. Terry starts his car and pursues Dan at 90 miles per hour, leaving 5 seconds after Dan passes him. Find a formula for the distance Terry has traveled from the police station t seconds after Dan leaves the vault. Graph this equation using your calculator and label it.

(j) Does Terry catch Dan? Explain, using your graph, how you know.

(k) We want to find out when Terry catches up to Dan. To do this, we'll find an equation describing how far Dan is in front of Terry.

i. How far is Dan in front of Terry 52 seconds after Dan leaves the vault?

ii. How about t seconds after Dan left the vault?

iii. This last expression is the equation of the distance Dan is in front of Terry. Graph this function. You may want to increase your viewing window to something like $-40 \leq t \leq 120$ by $-200 \leq y \leq 1000$.

iv. When does Terry catch Dan? What does this question have to do with roots of the function in iii.?

v. What is the significance of the lowest point on the graph of the function in iii.?

11. This problem is related to Example 2.5.7. Recall, we have these five functions:

$$\begin{aligned} f(x) &= x^2 \\ v(x) &= x+2 \\ d(x) &= 3x. \end{aligned} \qquad \qquad \begin{aligned} h(x) &= x-1 \\ r(x) &= -x \end{aligned}$$

Calculate these compositions:

(a) $v(r(d(h(f(x)))))$.

(b) $h(f(d(r(v(x)))))$.

(c) $h(f(v(r(d(x)))))$.