2.4 Composition

1. Compute the compositions $f(g(x))$, $f(f(x))$ and $g(f(x))$ in each case:

   (a) $f(x) = x^2, g(x) = x + 3$.
   (b) $f(x) = 1/x, g(x) = \sqrt{x}$.
   (c) $f(x) = 9x + 2, g(x) = \frac{1}{9}(x - 2)$.
   (d) $f(x) = 6x^2 + 5, g(x) = x - 4$.
   (e) $f(x) = 4x^3 - 3, g(x) = \sqrt[3]{2x + 6}$
   (f) $f(x) = 2x + 1, g(x) = x^3$.
   (g) $f(x) = 3, g(x) = 4x^2 + 2x + 1$.
   (h) $f(x) = 2x^3 - 5, g(x) = \sqrt[3]{\frac{x+5}{2}}$.

2. A car leaves Seattle heading east. The speed of the car in mph after m minutes is given by the function
$$C(m) = \frac{70m^2}{10 + m^2}.$$

   (a) Find a function $m = f(s)$ that converts seconds s into minutes m. Write out the formula for the new function $C(f(s))$; what does this function calculate?
   (b) Find a function $m = g(h)$ that converts hours h into minutes m. Write our the formula for the new function $C(g(h))$; what does this function calculate?

3. Write each of the following functions as a composition of two simpler functions: (There is more than one correct answer.)

   (a) $y = (x - 11)^5$.
   (b) $y = \sqrt[3]{1 + x^2}$.
   (c) $y = 2(x - 3)^5 - 5(x - 3)^2 + \frac{1}{2}(x - 3) + 11$.
   (d) $y = \frac{1}{x^2+3}$.
   (e) $y = \sqrt{\sqrt{x} + 1}$.
   (f) $y = 2 - \sqrt{5 - (3x - 1)^2}$.
   (g) $y = (3x - 5)^2 + 4(3x - 5)$.
   (h) Suppose $y = \sqrt{(4(x - 6))^2 - 8}$. Find simpler functions $f(x), g(x)$ and $h(x)$ so that $y = f(g(h(x)))$.

4. Let $y = f(z) = \sqrt{4 - z^2}$ and $z = g(x) = 2x + 3$. Compute the composition $y = f(g(x))$. Find the largest possible domain of x-values so that the composition $y = f(g(x))$ is defined.

5. The volume V of a sphere of radius r is given by the formula $V(r) = \frac{4}{3}\pi r^3$. A balloon in the shape of a sphere is being inflated with gas. Assume that the radius of the balloon is increasing at the constant rate of 2 inches per second, and is zero when $t = 0$.

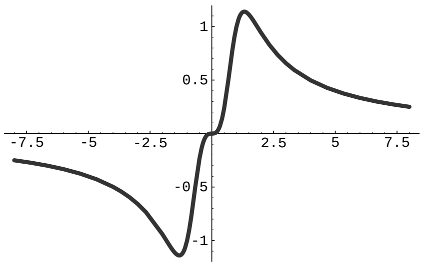
   (a) Find a formula for the volume V of the balloon as a function of time t.
   (b) Determine the volume of the balloon after 5 seconds.
   (c) Suppose that the balloon will burst when its volume is 10,000 cubic inches. At what time will the balloon burst?

(d) Find a formula for the surface area S of the balloon as a function of time t; recall the surface area formula for a sphere of radius r is $S(r) = 4\pi r^2$.

(e) Determine the surface area of the balloon after 6 seconds.

(f) What will be the surface area of the balloon when it bursts?

6. Pictured is the graph of the function $f(x) = \frac{2x^3}{1+x^4}$ on the domain $-7.5 \le x \le 7.5$.



In each of these cases, use your graphing capabilities (paper, software or calculator) to study the function graph and explicitly compute the corresponding equation for the curve:

(a) $f(x) + 1$,

(b) $f(x + 1)$,

(c) $2f(x + 1) + 1$,

(d) $f(\frac{x}{2})$,

(e) $2f(\frac{1}{2}(x + 1)) + 1$.

7. Suppose you have a function $y = f(x)$ such that the domain of $f(x)$ is $1 \le x \le 6$ and the range of $f(x)$ is $-3 \le y \le 5$.

(a) What is the domain of $f(2(x - 3))$?

(b) What is the range of $f(2(x - 3))$?

(c) What is the domain of $2f(x) - 3$?

(d) What is the range of $2f(x) - 3$?

(e) Can you find constants B and C so that the domain of $f(B(x - C))$ is $8 \le x \le 9$?

(f) Can you find constants A and D so that the range of $Af(x) + D$ is $0 \le y \le 1$?

8. Suppose you have a function $y = f(x)$ such that the domain of $f(x)$ is $-1 \le x \le 2$ and the range of $f(x)$ is $1 \le y \le \sqrt{10}$.

(a) What is the domain of $f(\frac{2}{3}(x + 1))$?

(b) What is the range of $f(\frac{2}{3}(x + 1))$?

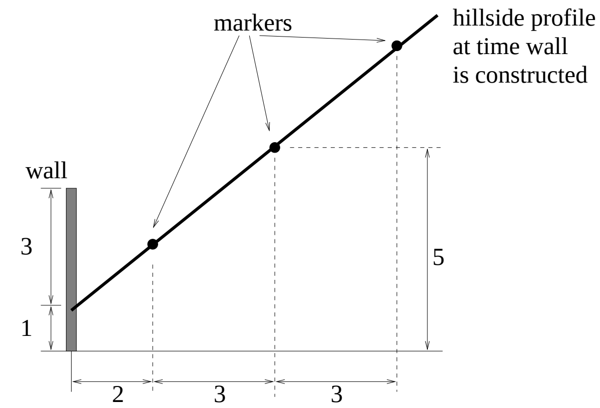
(c) What is the domain of $\frac{2}{3}f(x) - 3$?

(d) What is the range of $\frac{2}{3}f(x) - 3$?

(e) Can you find B and C so that the domain of $y = f(B(x - C))$ is $-1 \le x \le 0$?

(f) Can you find A and D so that the range of $Af(x) + D$ is $-1 \le y \le 1$?

9. A contractor has just built a retaining wall to hold back a sloping hillside. To monitor the movement of the slope the contractor places marker posts at the positions indicated in the picture; all dimensions are taken in units of meters.



(a) Find a function $y = f(x)$ that models the profile of the hillside.

(b) Assume that the hillside moves as time goes by and the profile is modeled by a function $g_n(x)$ after n years. If $n = 0$, then $g_0(x) = f(x)$. After one year, the profile is modeled by the function $g_1(x) = f(f(x))$. After two years, the profile is modeled by the function $g_2(x) = f(f(f(x)))$. After n years, it is modeled by the function $g_n(x) = f(f(\ldots f(x)\ldots))$, where we have composed the original function $n+1$ times. Find a formula for $g_n(x)$ that does not involve compositions. (Hint: To do this, start by writing out the formulas for $n = 1, 2, 3, 4$. You will see a pattern developing. To get the general formula, the following fact will be useful: Given a real number $0 < r < 1$ and a positive integer k,

$$1 + r + r^2 + r^3 + r^4 + \ldots + r^k = \frac{1 - r^{k+1}}{1 - r}.$$

Your final formula for $g_n(x)$ will involve both x and n.)

(c) Sketch the graphs of $g_n(x)$ for $n = 0, 1, 2, 3, 4, 5$ in the same coordinate system.

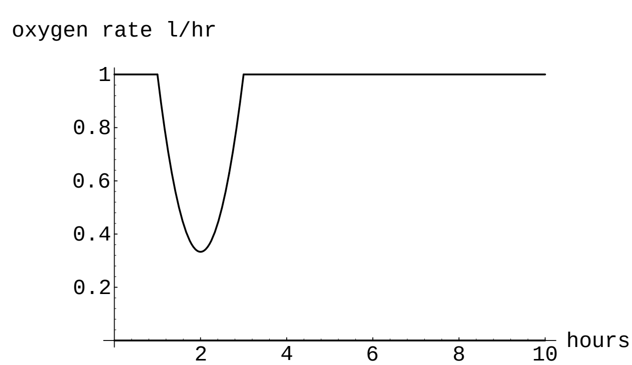
(d) What is happening to the marker posts?

(e) Estimate when the hillside will start to spill over the retaining wall.

10. Explain what each graph in Example 2.4.8 tells us. You don't need to use any explicit formulas; use the graphical analysis terminolgy from §2.2.

11. A plant is growing under a particular steady light source. If we apply a flash of high intensity green light at the time $t = 1$ and measure the oxygen output of the plant, we obtain the plot below and the mathematical model $f(t)$.

$$f(t) = \begin{cases} 1 & \text{if } t \le 1 \\ \frac{2}{3}t^2 - \frac{8}{3}t + 3 & \text{if } 1 \le t \le 3 \\ 1 & \text{if } 3 \le t \end{cases}$$



(a) Suppose instead we apply the flash of high intensity green light at the time $t = 2$. Verify that the mathematical model for this experiment is given by $f(g(t))$, where $g(t) = t - 1$. Sketch the graph modeling this experiment.

(b) Suppose you subject the plant to a flash of high intensity green light at the time $t = 2$ and at time $t = 5$. Sketch the graph modeling this experiment and find the corresponding multipart function. (The work in Example 2.4.8 will save some time.)

12. Let $f(x) = x^{1/3}$, $g(x) = x - 2$ and $h(x) = x + 2$. Take the domain of all these functions to be all real numbers. Find the formulas for $f(g(x)), g(f(x)), f(h(x))$ and $h(f(x))$. Discuss the a relationship between the graphs of these four functions. The graph of $f(x)$ is given below.