2.3 Quadratic Modeling

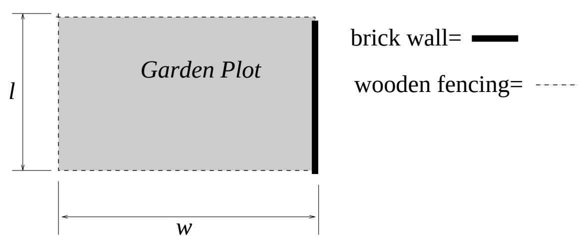
1. In Example 2.3.11 we claimed that the points $(0, 50)$, $(10, 80)$ and $(20, 200)$ do not lie on a common line. Give a convincing argument that this is true which uses a picture and an alternate argument that does not use a picture.

2. Write the following quadratic functions in vertex form, find the vertex and axis. Sketch a picture or compare with a plot from a graphing device:

   (a) $y = f(x) = 2x^2 - 16x + 41$.
   (b) $y = f(x) = 3x^2 - 15x - 77$.
   (c) $y = f(x) = x^2 - \frac{3}{7}x + 13$.
   (d) $y = f(x) = -0.2x^2 + \frac{1}{3}x - \frac{8}{5}$.
   (e) $y = f(x) = 2x^2$.
   (f) $y = f(x) = \frac{1}{100}x^2$.

3. The stopping distance d (in feet) of a certain car traveling at x miles per hour is approximated by the formula $d(x) = x + \frac{x^2}{20}$. At what speed will the stopping distance be approximately 110 feet? 34 feet? 8 feet?

4. You want to fence off a rectangular garden plot by putting a short brick wall along one edge and wooden fencing along the other three edges; see picture below. The brick wall will cost \$8 per linear foot, while the wooden fence costs \$2 per linear foot.

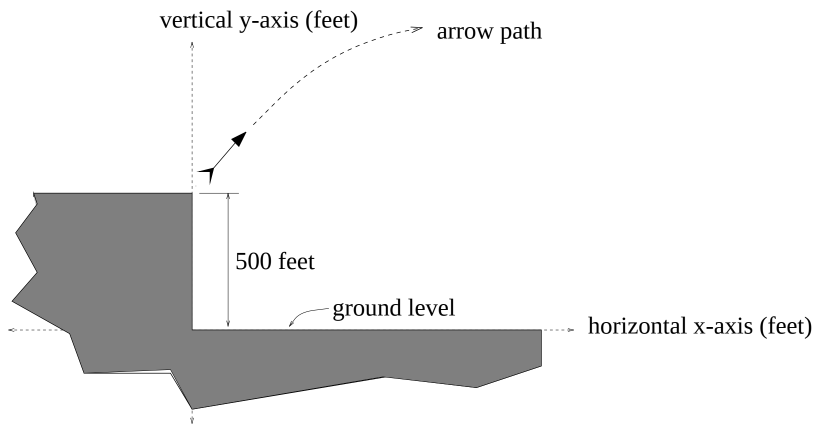


   (a) Find a function that gives the total cost of the materials in terms the variables l and w; ignore the thickness of the brick.

   (b) If you have only \$500 to spend on materials, what are the dimensions of the largest (biggest area) plot you can enclose?

5. An arrow is shot from the top of a 500 ft. high cliff at a slight upward angle; see picture below. Impose a coordinate system as in the figure.

   (a) If the vertical height of the arrow t seconds after firing is given by the function $s = y(t) = -16t^2 + 300t + 500$ feet, then what is the highest point reached by the arrow?
   (b) When does the arrow hit the ground?
   (c) When is the arrow 540 feet above ground level?
   (d) If the horizontal position of the arrow t seconds after firing is given by the function $s = x(t) = 400t$ feet, then where does the arrow hit the ground?

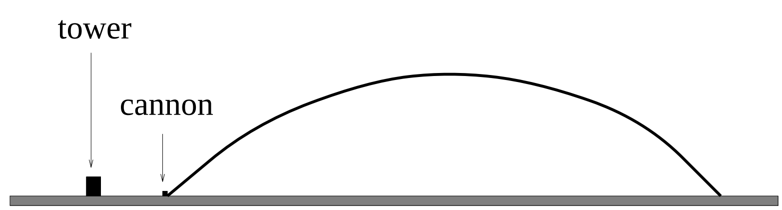


6. Calvin goes flying off the edge of a cliff (at a slight upward angle) in his wagon. His height above the ground at time t, where t is the time after leaving the ground, is given by

$$h(t) = -16t^2 + 16t + 25.$$

   (a) How high is the cliff?
   (b) What is the maximum height of the wagon above the ground?
   (c) When does the maximum height occur?

7. A cannon is located on level ground to the right of a spotting tower. A person in the spotting tower observes a cannon shot and knows the path of motion of the cannon ball is given by the graph of a quadratic function $q(x)$, where x is the horizontal distance from the observer to the cannon ball (in miles) and the ball height $q(x)$ is also in units of miles. During the observation, the person is able to take three measurements: 5 miles in front of the observer the ball is 2 miles high; 6 miles in front of the observer the ball is 3 miles high; 7 miles in front of the observer the ball is 2.5 miles high.



   (a) Model the path of the cannon ball.
   (b) Find the location of the cannon, relative to the observer.
   (c) What is the maximum height of the cannon ball and where does it achieve this height?
   (d) Where does the cannon ball strike the ground?
   (e) Where is the cannon ball at a position 800 feet in front of the cannon? (Recall, 1 mile = 5280 feet.)
   (f) Where is the cannon ball 560 feet high?

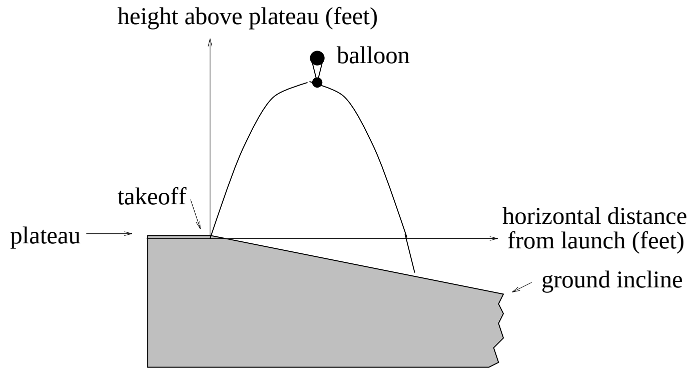
8. With 8 seconds left in the Rose Bowl game, the University of Washington Huskies are trailing the University of Michigan by a score of 21 to 23. The Dawgs are about to attempt a field goal. The ball is placed on the ground 43 yards in front of the goal post. The crossbar on the goal post is 15 feet high. Assume these three facts: (i) the path of the kicked ball is a parabola; (ii) the ball lands 150 feet in front of the kicker; (iii) the maximum height of the ball is 30 feet and this occurs midway between the tee and the landing point of the ball.

(a) Find the quadratic function describing the path of the kicked ball.

(b) Does the ball clear the goalpost?

(c) Where is the ball exactly 15 feet high?

9. A hot air balloon takes off from the edge of a plateau. Impose a coordinate system as pictured below and assume that the path the balloon follows the graph of the quadratic function $y = f(x) = -\frac{4}{2500}x^2 + \frac{4}{5}x$. The land drops at a constant incline from the plateau at the rate of 1 vertical foot for each 5 horizontal feet. Answer the following questions:



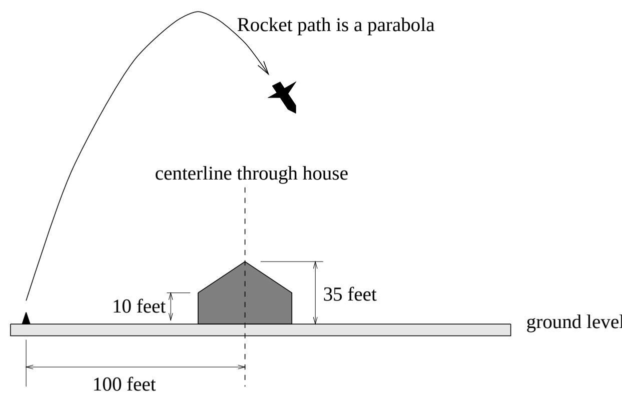
(a) What is the maximum height of the balloon above plateau level?

(b) What is the maximum height of the balloon above ground level?

(c) Where does the balloon land on the ground?

(d) Where is the balloon 50 feet above the ground?

10. A toy rocket is launched from a position 100 feet from the centerline of a house; see picture below. The highest point on the house roof is 35 feet above the ground and the roof slope is $\pm\frac{3}{5}$. The sides of the house are 10 feet high. Assume coordinates are imposed so that the x-axis coincides with ground level and the centerline through the house intersects the ground at the origin.



The path of the rocket is always the graph of a quadratic function, but the actual path will depend on the type and amount of fuel used. Answer these questions:

(a) If the path is given by the graph of $y = -\frac{1}{10}x^2 - 10x$, determine where the rocket lands, its maximum height above the ground and whether the rocket strikes the house. If the rocket strikes the house, determine exactly where it hits the house.

(b) If the path is given by the graph of $y = -\frac{1}{10}x^2 + 1000$, determine where the rocket lands, its maximum height above the ground and whether the rocket strikes the house. If the rocket strikes the house, determine exactly where it hits the house.

(c) If the path is given by the graph of $y = -\frac{1}{10}(x^2 + 90x - 1000)$, determine where the rocket lands, its maximum height above the ground and whether the rocket strikes the house. If the rocket strikes the house, determine exactly where it hits the house.

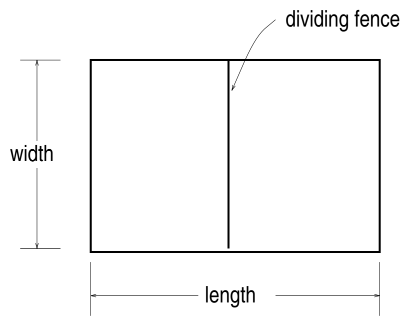
(d) If the path is given by the graph of $y = -\frac{1}{50}(x^2 + 150x + 5000)$, determine where the rocket lands, its maximum height above the ground and whether the rocket strikes the house. If the rocket strikes the house, determine exactly where it hits the house.

(e) Study the rocket paths in (a)-(d) in a common coordinate system using a graphing device.

11. Steve likes to entertain friends at parties with "wire tricks". Suppose he takes a piece of wire 60 inches long and cuts it into two pieces. Steve takes the first piece of wire and bends it into the shape of a perfect circle. He then proceeds to bend the second piece of wire into the shape of a perfect square. Where should Steve cut the wire so that the total area of the circle and square combined is as small as possible? What is this minimal area?

12. In 2.3.6, we stated that the vertex of the graph of a quadratic function $f(x) = ax^2 + bx + c$ has the form $(\frac{-b}{2a}, f(\frac{-b}{2a}))$. Express $f(\frac{-b}{2a})$ in terms of a, b and c.

13. Tim and Michael quit graduate school and buy farmland in New Jersey. They decide to equally split a small rectangular plot, since Tim wants to grow alfalfa sprouts and Michael wants to breed llamas. They want to use a length of fencing to place a border around the entire plot and to make a divider to keep the livestock out of the sprouts.



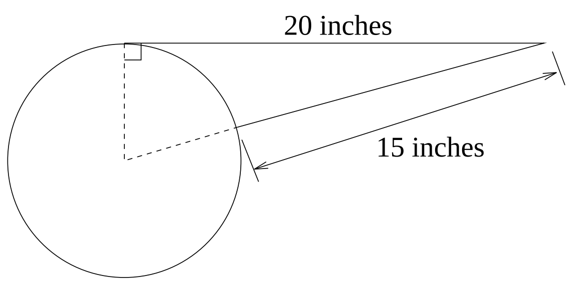
(a) Due to budgetary constraints they can only afford to get 540 feet of fencing. Write an equation relating length and width in the picture above.

(b) What are the dimensions of the plot to maximize the area given the constraint in a.?

14. You are stranded on a desert island with only paper, pencil, a straight-edge and a compass (no ruler). To kill time, you decide to construct points on the graph of $y = x^2$; see exercise 2.1.26. How do you do this?

15. Sketch the graph of $y = x^2 - 2x + 2$. For various constants b, consider the intersection with the graph of $y = x + b$. Via both pictures and equations, what happens when $b > -1/4$, $b = -1/4$ and $b < -1/4$?

16. Find the radius of the circle pictured:



17.* With 8 seconds left in the Rose Bowl game, the University of Washington Huskies are trailing the University of Michigan by a score of 22 to 23. The Dawgs are about to attempt a field goal. The ball is placed on the ground 43 yards in front of the goal post. The crossbar on the goal post is 15 feet high. Assume these three facts: (i) the path of the kicked ball is a parabola; (ii) the ball clears the crossbar by 18 inches; (iii) the maximum height of the ball is 50 feet and this occurs *somewhere* between the tee and the landing point of the ball.

(a) Find the quadratic function describing the path of the kicked ball. (Hint: Impose a coordinate system with the kicker as the origin. Use the "vertex form" $f(x) = a(x - h)^2 + k$ for a quadratic function. Obtain an equation involving the unknowns a and h, based on the kickers position. Obtain another equation based on the information about clearing the bar. Solve for h.)

(b) Where does the ball land?

(c) Where is the ball exactly 15 feet high?

(d) Suppose you change the given information slightly. Assume you are given (i), (ii) and the fact that the maximum height of the ball is 50 feet (but you don't know where the maximum occurs). What could you say about the path of the ball?

18. (Magical Marble Toss) A marble is thrown into the air from a ledge so that it starts $10\,ft$ above ground. The path of the marble is given by the graph of $y = -\frac{1}{22}x^2 + 4x + 10$ in the pictured coordinate system. Suspended magically in the air are bowls of paint. Each bowl is $2\,ft$ wide at its mouth. If the marble lands in a bowl, it will become the color of the paint in the bowl and magically pass through the bowl, continuing on its way. The location of each bowl and the color of its paint is given in the table. The marble starts out clear. What color is the marble when it hits the ground?

Paint	Coordinates Bowl Left Rim
White	(44, 100)
Yellow	(70, 67)
Blue	(73, 54)
Green	(89, 1)

19. A sailing club charges its members \$100 per year in dues. The fee for every member is reduced by \$1 for each member in excess of 60. Thus, for example, if the club had 65 members, the fee for each of those 65 members would be reduced by \$5 ($= \$1 \times (65-60)$), and so each member would pay \$100-\$5=\$95 per year.

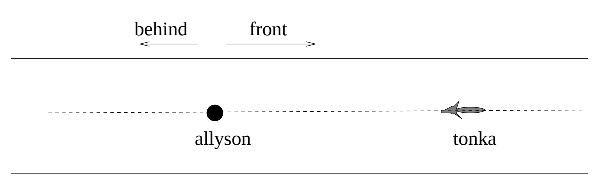
(a) If x represents the number of members in the club, find a formula for the total dues revenue which is valid when $x > 60$.

(b) What number of members would maximize the dues revenue?

20. (a) Suppose that θ is a quadratic function in the variable t; write down a formula for θ.

(b) Suppose that t is a quadratic function in the variable θ; write down a formula for t.

21. Allyson is playing with her dog *Tonka* who starts 20 feet in front of her and runs back and forth along a straight line, as pictured. The function $d(t) = \frac{15}{8}(t-4)^2 - 10$ describes Tonka's distance in front of Allyson at time t seconds; note that a negative value for this distance means that Tonka is behind Allyson.



TOP VIEW- not to scale

(a) Sketch an accurate plot of the graph of $y = d(t)$ for the first 10 seconds; feel free to use a graphing device.

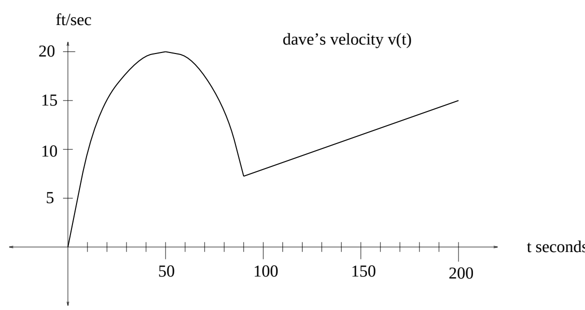
(b) When is Tonka located right next to Allyson? Interpret your answer in terms of the graph in a.

(c) During the first 10 seconds, how much time does Tonka spend behind Allyson? Interpret your answer in terms of the graph in a.

(d) During the first 10 seconds, find where and when Tonka is located the farthest behind Allyson. Interpret your answer in terms of the graph in a.

(e) During the first 10 seconds, find where and when Tonka is located the farthest in front of Allyson. Interpret your answer in terms of the graph in a.

22. Dave starts off on a run from the IMA. Assume he runs in a straight line for the first 200 seconds. Here is a picture of the graph of his velocity $v(t)$ at time t seconds; we will use distance units of FEET and time units of SECONDS. Here is the actual formula for Dave's velocity:



$$v(t) = \begin{cases} \frac{-1}{125}t^2 + \frac{8}{10}t, & 0 \le t \le 90 \\ 0.07091t - 0.818, & 90 \le t \le 200 \end{cases}$$

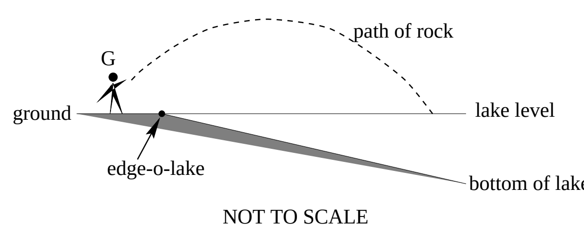
(a) When does Dave have maximum velocity and what is his maximum velocity? What is his "pace" at this time in units of "minutes/mile"?

(b) During the first 200 seconds, how much time does Dave spend running at a speed of at least 12 ft/sec?

23. Complete the following table:

Equation	a	b	c	vertex	point	point	point	root	root
	-1	0	2						
	-1	2	-6						
$-x^2 + 2x + 4$									
	-3			(2,1)					
$-2(x+1)^2 - 3$									
	1							1	5
	1				(1,5)	(5,5)			
					(3,11)	(1,5)	(-1,3)		
					(2,2)	(3,5)		1	
				(-3,4)	(1,2)				

24. Georgia is standing two feet from the edge of a lake, as pictured. The bottom of the lake gently slopes downward from the edge with a slope of -1/4. She throws a rock toward the lake. If the rock is x feet in front of the the edge of the lake, assume it's height above lake level is given by the function $y = -0.02x^2 + 0.92x + 7.92$ feet.



(a) How should you impose coordinates and why?

(b) What is the maximum height of the rock above lake level?

(c) Where does the rock hit the water?

(d) How high is the rock when Georgia releases it?

(e) What is the maximum height of the rock above the bottom of the lake?

25.* In this exercise, we introduce a new technique for solving an old type of problem. At this stage, the old approach will probably seem more comfortable, but we will see the new approach occur quite often in Chapter 4. Here is the problem we want to solve in two different ways: Find the point on the line $y = 3x - 11$ closest to the point $(-1, 5)$ and the distance to this closest point.

(a) Sketch a picture of the situation.

(b) (Old Approach) Solve the problem by constructing a line perpendicular to the given line which passes through $(-1, 5)$, find the intersection of the two lines, etc.

(c) (New Approach) Let $P = (x, y)$ be a typical point on the graph of $y = 3x - 11$. Form a NEW function $y = d(x)$, which computes the distance from $(-1, 5)$ to P. We call this the *distance function*. The new idea is to "minimize the distance function". Here are the steps:

i. What is the formula for $y = d(x)$?

ii. Form a second NEW function $y = (d(x))^2$. We call this the *distance squared function*. What is the formula for $y = (d(x))^2$?

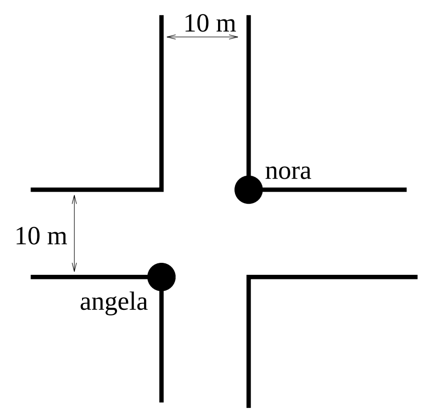
iii. Use a graphing device to plot the graphs of $y = d(x)$ and $y = (d(x))^2$ simultaneously. What qualitative feature do these two graphs have in common?

iv. Find the coordinates of the lowest point on the graph of the function $y = (d(x))^2$.

v. Find the coordinates of the lowest point on the graph of the function $y = d(x)$.

vi. Find the point on line $y = 3x - 11$ closest to the point $(-1, 5)$ and the distance to this closest point.

26. Angela and Nora are standing on opposite corners of an intersection. The roads are both 10 meters wide. Nora doesn't move, but Angela moves toward Nora along two different paths. Impose coordinates with Angela initially at the origin. We will study where Angela is closest to Nora in each case.



(a) Angela moves toward Nora along the line $y = \frac{1}{2}x$. Let d be the distance from Nora to a typical point (x, y) on Angela's path.

i. Find formulas for d and d^2 as functions of x.

ii. Use a graphing device to sketch the graph of both of these functions. Label each graph and find the minimum value of d^2.

iii. Where is Angela when she is closest to Nora?

(b) Angela moves toward Nora along the graph of $y = 0.3(x-1)^2 - 0.3$. Let d be the distance from Nora to a typical point (x, y) on Angela's path.

i. Find a formula for d as a function of x.

ii. Use a graphing device to sketch the graph of d. Find the minimum value of d using your graphing device.

iii. Where is Angela when she is closest to Nora?

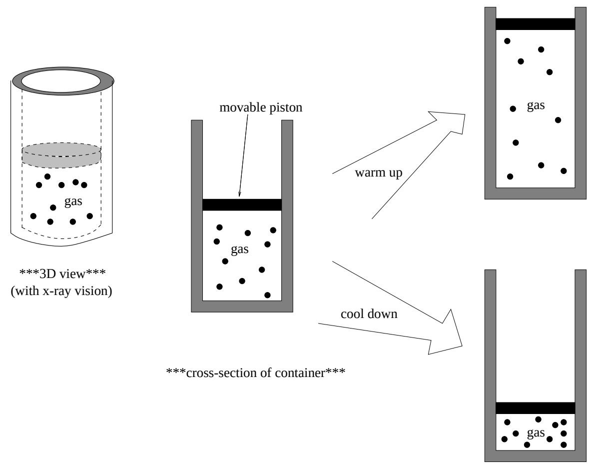
27. This exercise is all about modeling some experimental data from a chemistry lab. It illustrates *how* linear regression modeling of data works. We will need to measure **temperature** and **volume**, so we need to agree on the units to use:

- **volume** will be measured in liters and denoted ℓ.
- **temperature** will be measured in Kelvin degrees, denoted by ^oK, where

$$t \text{ degrees Kelvin} = (t - 273) \text{ degrees Celsius}$$

You walk into your chemistry lab and sitting in front of you is a container of gas with a movable piston on the top (as pictured, left below). In this experiment, the amount of gas in the chamber (i.e. the number of actual gas molecules in the chamber) is not going to change. We want to study the relationship between the **temperature** of the gas and the **volume** of the gas chamber. After fooling around a bit you would make two experimental observations:

- If you apply some heat to the chamber, the gas heats up and the piston rises to create a bigger volume.
- Conversely, if you cool the chamber, then the gas cools down and the piston lowers to create a smaller volume.



We are going to assume this **gas law** for the purposes of this problem:

> **SPECIAL GAS LAW.** For the container of gas given, the temperature t (in ^oK) and the volume v (in ℓ) are proportional.

Two quantities are proportional if their ratio is a constant, so this means that there is a constant, let's call it R, so that

$$v = Rt$$

Notice, this is says that volume is a linear function of temperature; i.e. $v(t) = v = Rt$.

The goal is to take some experimental data and come up with our prediction of the constant R, which is called the *gas constant*.

(a) In the lab, you gather this experimental data. Plot this data in a coordinate system. Label the vertical axis as volume (running from 0 to 100 ℓ) and the horizontal axis as temperature (running from 0^oK to 1200^oK).

t (^{o}K)	v (liters)
200	16
400	20
600	64
1200	92

(b) By the Special Gas Law, we know that the graph of the function $v = Rt$ is a line. On on the graph from a., draw in all possible linear models that pass through pairs of data points. Does any one of these lines model the Special Gas Law relationship: $v = Rt$?

(c) Now, we want to try and find the "line of best fit" for the given data. This should approximate the Special Gas Law line $v = Rt$. To do this, start by drawing a new coordinate system and plotting the four data points. Draw in a line ℓ which you guess to be *the best fit for this data*. At this point, the line ℓ is only a preliminary guess, but we need the line in front of us to analyze it. The equation of ℓ will be $v = Rt$, for some R (and finding R for the best fit is the whole ballgame).

i. In your picture, sketch the four little vertical line segments from the four data points up (or down) to the line ℓ.

ii. Compute and label the length of each of these little vertical line segments IN TERMS OF R.

iii. Finally, add up the squares of these four lengths and call it E. You should get an expression for E that ONLY involves the variable R:

$$\begin{aligned} E &= (\text{length of vertical segment } \#1)^2 + (\text{length of vertical segment } \#2)^2 + \\ &\quad (\text{length of vertical segment } \#3)^2 + (\text{length of vertical segment } \#4)^2 \end{aligned}$$

(d) We will call the expression E in c(iii) the *error term*. Explain why that terminology makes sense in terms of the picture.

(e) Suppose we want to minimize the error term E. Use what you know about quadratic functions to find the smallest value of E AND the R for which this occurs.

(f) Let (R, E) be the vertex computed in e. Refer to R as the *gas constant*. Return to your original graph in a. and sketch in the line $v = Rt$. How closely does your experimental data fit this "line of best fit"?

(g) Can you find a quadratic function whose graph passes through the four data points?

(h) Here is a third degree polynomial; we call this a *cubic function*. Plot this on your graphing calculator and sketch it into your first graph. Use the window $0 \le x \le 1200$ by $0 \le y \le 110$.

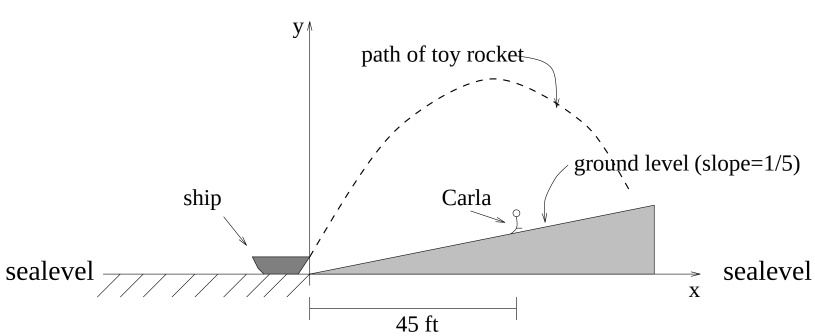
$$p(x) = -7.16667 \times 10^{-7}x^3 + 0.00136x^2 - 0.595333x + 86.4.$$

Give one argument that $p(x)$ is a good fit to the given data. Give another argument that $p(x)$ is a bad fit.

28. A ship is docked at the edge of a long hill which has slope equal to $\frac{1}{5}$. A toy rocket is launched from the front of the ship, and its path is along the parabola

$$y = -\frac{1}{40}x^2 + 2x + 2,$$

where the coordinate system is as in the picture and all distances are given in feet. Carla, the Canadian cop, is standing at the location in the picture.



(a) What is the elevation of the rocket above sealevel the instant it is launched?

(b) What is the maximum height of the toy rocket above sea level?

(c) Find the equation of the line which represents the ground level in the coordinate system above.

(d) How high is the toy rocket **above the ground** when it flies over Carla? Notice that the ground slopes upward, so ground level is **not** the x-axis.

(e) Where does the toy rocket hit the **ground**?

(f) Find the equation, $h(x)$, which gives the height of the toy rocket **above the ground**.

(g) Find the maximum height of the toy rocket **above the ground.**

29. You have decided to write and sell a booklet called *The Math 120 Student Survival Guide*. A friend has done some marketing analysis for you. Her study shows that if you charge \$12, you will sell 500 copies. But, for each \$0.25 you lower the price, sales will increase by an additional 40 copies. (For example, if you charge \$11.75, you will sell 540 copies; if you charge \$11.50, you will sell 580 copies, etc.) Finally, your startup costs to produce the booklet (things like a computer, printer, etc.) will be \$2800 and the production cost is \$3 **per book.**

(a) What sales price for the booklet will maximize your profit? (Hint: Your profit is computed by subtracting startup and production costs from the income you make on the total sales of the booklet.)

(b) What will be the total sales to maximize profit?

(c) What will be the maximum profit?

30. As in Example 2.3.2, describe a sequence of geometric operations leading from the graph of $y = x^2$ to the graph of $y = f(x) = -3(x+1)^2 - 2$. Sketch the graphs of $y = x^2$ and each of the graphs obtained by your sequence of geometric operations.