2.1 Functions and Graphs

1. In each of (a)-(h) decide which equations establish a function relationship between the independent variable x and the dependent variable y. Write out the rule for any functions.

   (a) $y = x + 1$

   (b) $2x - 3y = 5$

   (c) $x - 4 = 0$

   (d) $y + 2 = 7$

   e. $xy = 4$

   f. $x^2y = 4$

   g. $xy^2 = 4$

   h. $y - \sqrt{x^2+1} = 0$

2. This problem deals with the "mechanical aspects" of working with the rule of a function. For each of the functions listed in (a)-(e), calculate:

   (i) $f(0)=$

   (ii) $f(-2)=$

   (iii) $f(a)=$

   (iv) $f(x+3)=$

   (v) $f(\sqrt{t})=$

   (vi) $f(f(x))=$

   (vii) $f(\heartsuit)=$

   (viii) $f(\heartsuit + \triangle)=$

   (a) The function $f(x) = \frac{1}{2}(x-3)$ on the domain of all real numbers.
   (b) The function $f(x) = 2x^2 - 6x$ on the domain of all real numbers.
   (c) The function $f(x) = x^2 + x + 1$ on the domain of all real numbers.
   (d) The function $f(x) = 4\pi^2$ on the domain of all real numbers.
   (e) The function $f(x) = x\sqrt{x^2+4}$ on the domain of all real numbers.

3. (a) Solve the equation

$$\frac{x^2}{9} + 4(y+1)^2 = 1$$

   for y in terms of x. You should get two different functions of x. Describe the largest possible domain of each function.

   (b) Let a and b be positive constants. Solve the equation

$$\left(\frac{x+1}{a}\right)^2 + \left(\frac{y-2}{b}\right)^2 = 1$$

   for y in terms of x. You should get two different functions of x. Describe the largest possible domain of each function.

   (c) Solve the equation

$$\frac{x^2}{9} - 4(y+1)^2 = 1$$

   for y in terms of x. You should get two different functions of x. Describe the largest possible domain of each function.

   (d) Let a and b be positive constants. Solve the equation

$$\left(\frac{x-1}{a}\right)^2 - \left(\frac{y+4}{b}\right)^2 = 1$$

   for y in terms of x. You should get two different functions of x. Describe the largest possible domain of each function.

(e) Use a graphing device to sketch the graphs of each of the functions you found in parts (a)-(d).

4. For each of the following functions, find the expression for

$$\frac{f(x+h)-f(x)}{h}.$$

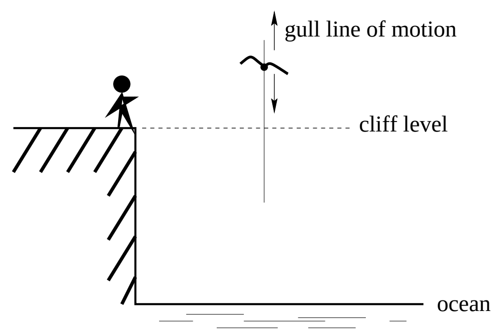
Simplify each of your expressions so that there is no h in the denominator.

(a) $f(x) = x^2 - 2x$.
(b) $f(x) = 2x + 3$
(c) $f(x) = x^2 - 3$
(d) $f(x) = 4 - x^2$
(e) $f(x) = -\pi x^2 - \pi^2$
(f) $f(x) = \sqrt{x-1}$. (Hint: Rationalize the numerator)

5. This problem builds on the "seagull analysis" at the start of this section and introduces *average velocity*. Here is the situation:

Imagine you are standing high atop an oceanside cliff and spot a seagull hovering in the air-current. Assume the gull moves up and down along a vertical line of motion and it's height above cliff level at time t seconds is given by the function

$$s = h(t) = \frac{15}{8}(t-4)^2 - 10.$$



The graph of the function $s = h(t)$ is given in Example 2.1.5 of the text.

(a) Let P be the location of the gull at time $t = 8$ seconds. Compute the coordinates of P and indicate this in the graph in Example 2.1.5.

(b) Let Q be the location of the gull at time $t = 4$ seconds. Compute the coordinates of Q and indicate this in the graph in Example 2.1.5. Draw the line through P and Q in your picture. In addition, compute the *average velocity* from 4 to 8 seconds by the formula: v_{ave}= slope of the line through P and Q.

(c) Repeat step (b) for time $t = 6, 7, 7.9, 7.99, 7.999$. It might be best to organize your data in a table. Do all of these average velocities appear to be getting close to some number?

6. In each of a.-f. below, an object is described which establishes a relationship between two variables. Identify the variables, decide which relationships are functions and in those cases identify the independent and dependent variables.

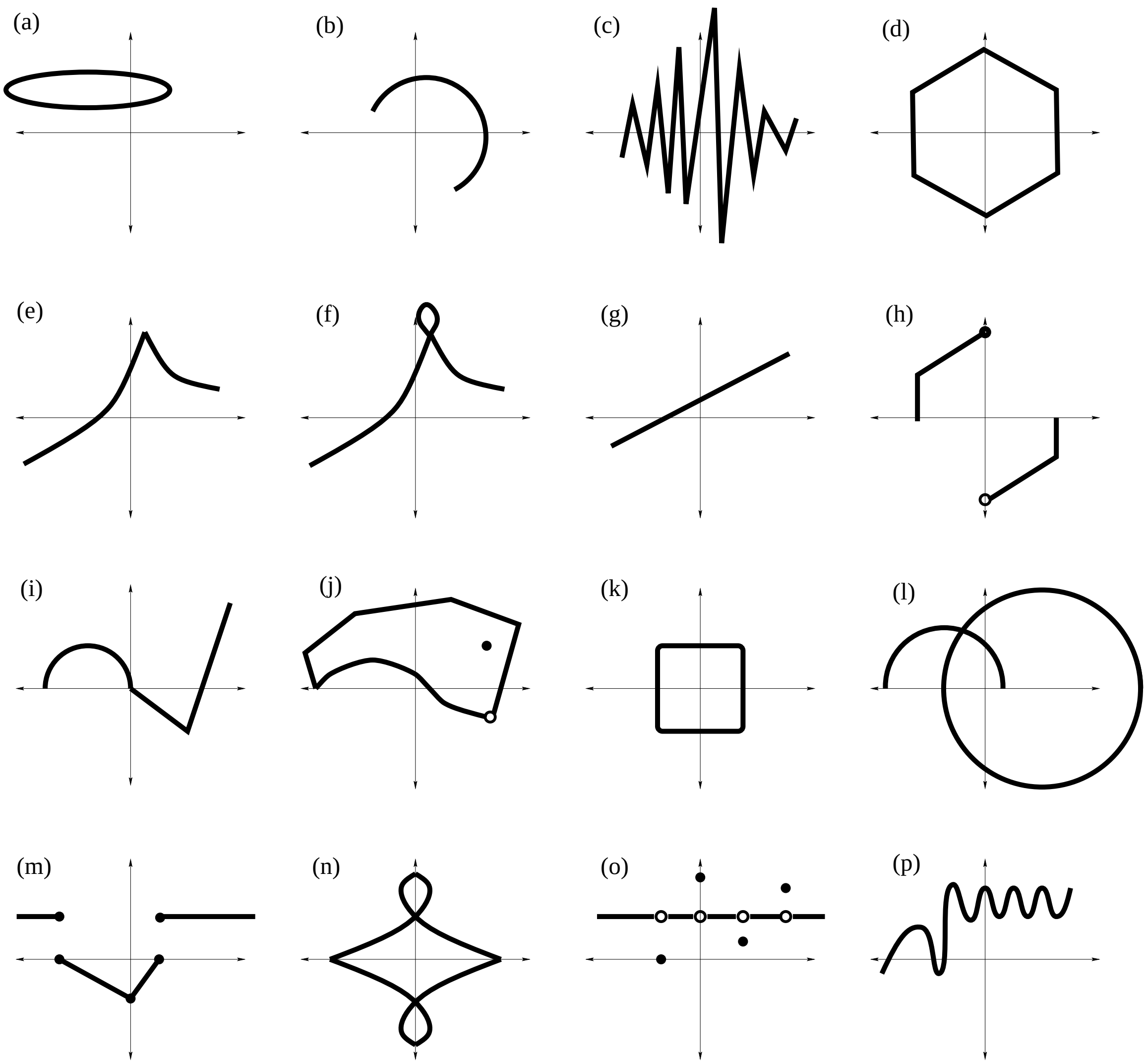
(a) An itemized receipt from the University Book Store.
(b) The index to this text.
(c) A bathroom scale.
(d) A radio dial.
(e) A phone book.

(f) Your math professor's grade book.

7. Sketch a reasonable graph for each of the following functions. Specify a reasonable domain and range and state any assumptions you are making. Finally, describe the largest and smallest values of your function.

   (a) Height of a person depending on age.
   (b) Height of the top of your head as you jump on a pogo stick.
   (c) The amount of postage you must put on a first class letter, depending on the weight of the letter.
   (d) Distance of your big toe from the ground as you ride your bike for 10 minutes.
   (e) The amount of electricity used, in kilowatt hours, by a household over a typical day.
   (f) The number of leaves on a tree related to the day of the year.
   (g) The temperature of the center of a cake that is put into the oven at 350^o related to time.
   (h) You pour some popcorn into a popper and turn it on. The number of pops per second related to the amount of time the popper has been turned on.
   (i) Your height above the water level in a swimming pool after you dive off the high board.

8. Which of these curves represent the graph of function? If the curve is not the graph of a function, describe what goes wrong and how you might "fix it". When you describe how to "fix" the graph, you are allowed to cut the curve into a finite number of pieces and study each piece individually. Many of these problems have more than one correct answer.



9. Complete this table:

part	Function	$f(0)$	$f(1)$	$f(a)$	$f(a+1)$	$f(\frac{z}{2}+1)$	$f(x^2)$
a	$y = 4x - 7$						
b	$y = x^2 + x + 1$						
c	$y = \pi^3$						
d	$y = x$						
e	$y = \frac{1}{x+8}$						
f	$y = \sqrt{x}$						

10. Dave leaves his office in Padelford Hall on his way to teach in Gould Hall. Below are several different scenarios. In each case, sketch a plausible (reasonable) graph of the function $s = d(t)$

which keeps track of Dave's distance s from Padelford Hall at time t. Take distance units to be "feet" and time units to be "minutes". Assume Dave's path to Gould Hall is along a straight line which is 2400 feet long.

(a) Dave leaves Padelford Hall and walks at a constant speed until he reaches Gould Hall 10 minutes later.

(b) Dave leaves Padelford Hall and walks at a constant speed. It takes him 6 minutes to reach the half-way point. Then he gets confused and stops for 1 minute. He then continues on to Gould Hall at the same constant speed he had when he originally left Padelford Hall.

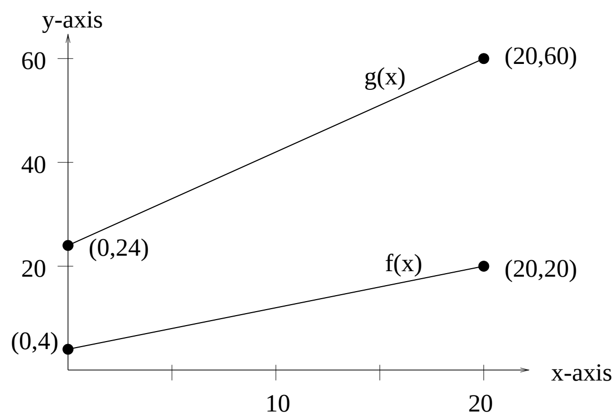
(c) Dave leaves Padelford Hall and walks at a constant speed. It takes him 6 minutes to reach the half-way point. Then he gets confused and stops for 1 minute to figure out where he is. Dave then continues on to Gould Hall at twice the constant speed he had when he originally left Padelford Hall.

(d) Dave leaves Padelford Hall and walks at a constant speed. It takes him 6 minutes to reach the half-way point. Dave gets confused and stops for 1 minute to figure out where he is. Dave is totally lost, so he simply heads back to his office, walking the same constant speed he had when he originally left Padelford Hall.

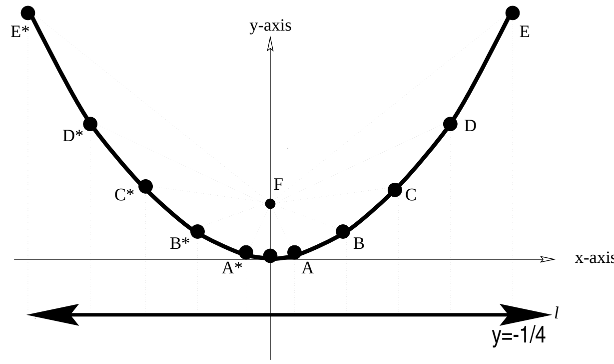
(e) Dave leaves Padelford heading for Gould Hall at the same instant Angela leaves Gould Hall heading for Padelford Hall. Both walk at a constant speed, but Angela walks twice as fast as Dave. Indicate a plot of "distance from padelford" vs. "time" for both Angela and Dave.

(f) Suppose you want to sketch the graph of a *new* function $s = g(t)$ that keeps track of Dave's distance s from Gould Hall at time t. How would your graphs change in a.-e.?

11. Here are the graphs of two linear functions on the domain $0 \leq x \leq 20$. Find the formula for each of the rules $y = f(x)$ and $y = g(x)$. Find the formula for a NEW function $v(x)$ that calculates the vertical distance between the two lines at x. Explain in terms of the picture what $v(x)$ is calculating. What is $v(5)$? What is $v(20)$? What are the smallest and largest values of $v(x)$ on the domain $0 \leq x \leq 20$?



12. Mark the point $F = (0, \frac{1}{4})$ on the y-axis and draw the horizontal line ℓ which is the graph of $y = -\frac{1}{4}$. The set of all points in the xy-plane equidistant from F and ℓ is a *parabola* opening upward, as pictured below. For example, we marked points $A, A^*, B, B^*, C, C^*, D, D^*, E, E^*$ which are equidistant from F and ℓ; the corresponding distances are indicated by dotted lines. The point $(0, 0)$ is the vertex and the vertical line $x = 0$ is the axis of symmetry.



(a) Let $P = (x, y)$ be a typical point on this parabola and Q the point on ℓ closest to P; this means you draw a vertical line ℓ^* through P and then Q is the point where ℓ^* intersects ℓ. Sketch a picture of this situation. What are the coordinates of Q?

(b) Use the distance formula and the fact $\overline{FP} = \overline{PQ}$ to show that the parabola is the graph of $y = x^2$.

13. Here are three tables of data. Sketch the graph of a function that matches each set of data.

x	y
10	50
9.5	40
8.9	30
8.4	20
7.7	10
6.3	-10
5.5	-20
4.5	-30
3.2	-40
0	-50

x	y
10	50
9	40
8	30
7	20
6	10
4	-10
3	-20
2	-30
1	-40
0	-50

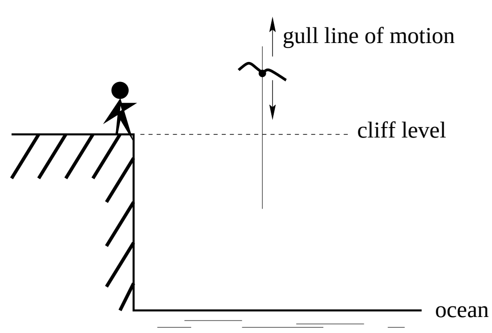
x	y
10	50
6.8	40
5.5	30
4.5	20
3.7	10
2.2	-10
1.6	-20
1	-30
0.5	-40
0	-50

14. The U.S. Justice Department is investigating a Utah based software distributor advertising illegal copies of *Microsoft Excel* spreadsheet software, priced at \$20 per copy. An undercover investigation locates documents showing the expenses involved in producing x illegal copies of the software depend on customer demand and are given by the function $y = e(x) = 0.05(x - 60)^2 - 5(x - 60) + 1000$.

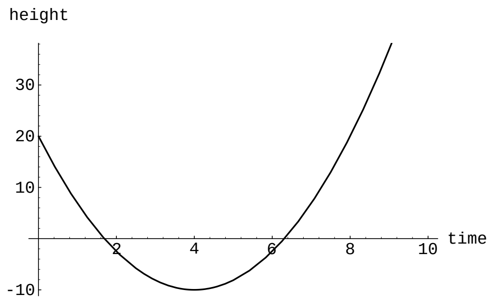
(a) Using a graphing device, sketch the graph of this function.

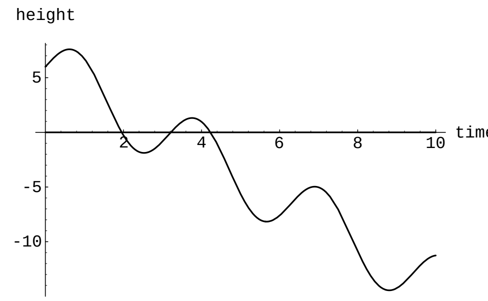
(b) When will the distributor realize a profit or loss? Estimate when profit is maximized. Find the break-even points. Should Bill Gates be concerned?

15. Shannon is standing high atop an oceanside cliff and spots seagulls hovering in the air-current. Assume a gull moves up and down along a vertical line of motion, as indicated. The questions below deal with plots of functions giving the height of a gull above cliff level after t seconds.

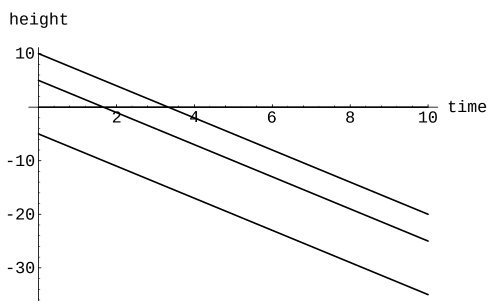


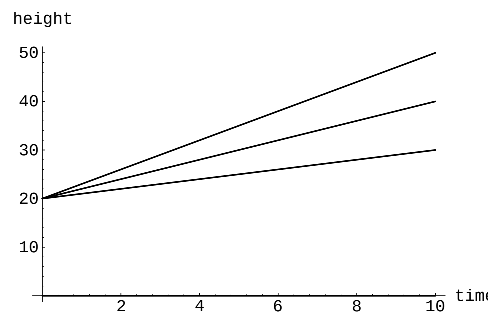
(a) Describe the motion of a gull given this two plots:



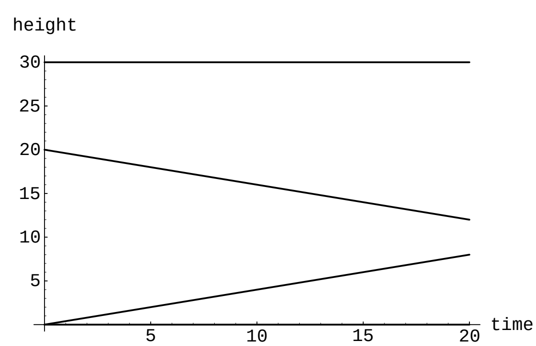


(b) Each plot simultaneously describes the motion of three different gulls. Describe what is happening in each scenario and what is the *vertical speed* of each gull?

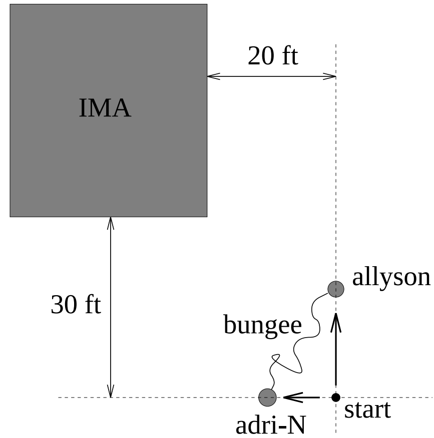




(c) Here you observe three different gulls simultaneously. What is happening in each case? Which gull has the largest and smallest vertical speed?



16. Allyson and Adrienne have decided to connect their ankles with a *bungee cord*; one end is tied to each persons ankle. The cord is 30 feet long, but can stretch up to 90 feet. They both start from the same location. Allyson moves 10 ft/sec and Adrienne moves 8 ft/sec in the directions indicated. In this question, neither girl stops moving until the bungee reaches is maximum length.



(a) Sketch an accurate picture of the situation for each of these three times: $t = 5, 5.5$ and 6 seconds. In each picture, make sure to label the locations of Allyson and Adriene; also, compute the length of the bungee cord.

(b) Find a function $b(t)$ that computes the length of the bungee cord at time t seconds. (Hint: This is a multipart function.)

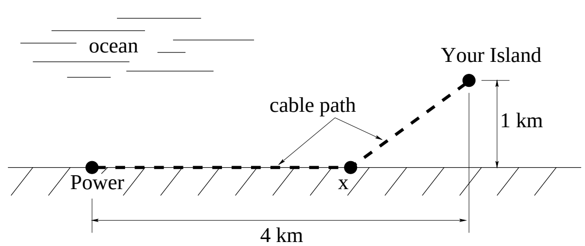
(c) Use a graphing device to sketch the graph of $b(t)$.

(d) Where are the girls located when the bungee reaches its maximum length?

(e) Recall the results in Problem 1.2.8(c). Find a function that computes the shortest distance from the corner of the building to the tightly stretched bungee at time t seconds.

17. The number of chemistry majors in an introductory chemistry class changes as function of the day t of the term according to the function: $c(t) = \frac{100((0.02t)^2+0.02t+1)}{(0.02t)^2+1}$. In the same class, the number of zoology majors is given by the function: $z(t) = \frac{100(3(0.02t)^2+1)}{(0.02t)^2+1}$. Use a graphing device to plot the graph of each function and describe what is happening over the course of the term. When (if ever) does the number of chemistry majors equal the number of zoology majors?

18. After winning the lottery, you decide to buy your own island. The island is located 1 km offshore from a straight portion of the mainland. You need to get power out to the island so you can listen to your extensive CD collection. A power sub-station is located 4 km from your island's nearest location to the shore. It costs \$50,000 per km to lay a cable in the water and \$30,000 per km to lay a cable over the land.



(a) Explain why we might as well assume that the cable follows the path indicated in the picture. In other words, explain why the path consists of two line segments, rather than a weird curved path. Also, as in the picture, explain why it is OK to assume the cable reaches shore to the right of the powerstation and the left of the island.

(b) Let x be the location downshore from the powerstation where the cable reaches the land. Find a function in the variable x that computes the cost to lay a cable out to your island.

(c) Use a graphing device to approximate the smallest cost to run power to your island

19. Nicole has decided to entertain party guests by flipping a coin. It is possible to give a function that tells the probability she will get exactly k heads in 10 tosses. To write down the formula

for this function, we need some special notation that comes up a lot in probability. We need the *factorial* notation: $k! = k \times (k-1) \times (k-2) \times (k-3) \times \ldots \times 2 \times 1$. For example, $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$, $4! = 4 \times 3 \times 2 \times 1 = 24$, etc. We DEFINE $0! = 1$. Now, the probability she will get exactly k heads in 10 tosses is given by this function:

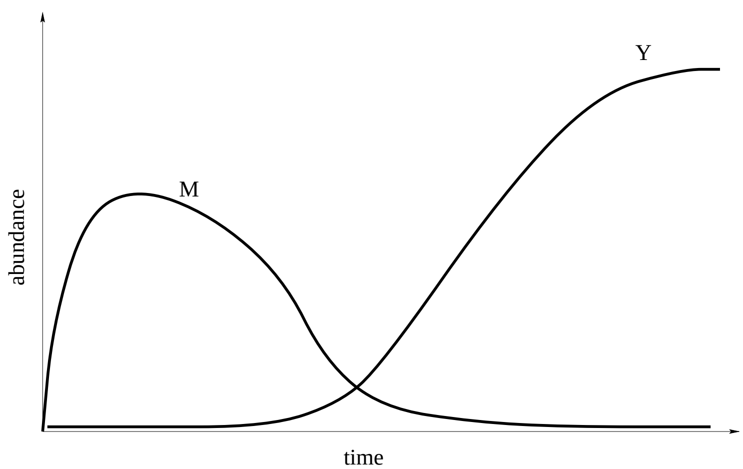
$$H(k) = \frac{10!}{k!(10-k)!2^{10}},$$

where $k = 0, 1, \ldots, 10$. Notice, the domain of this function is just the set of 11 numbers $\{0, 1, 2, \ldots, 10\}$.

(a) Plot the data points $(k, H(k))$ corresponding to the eleven numbers in the domain. Make sure to label your coordinate axes and explain the units you are using.

(b) What is the most likely number of heads Nicole will get in 10 tosses and how likely is it?

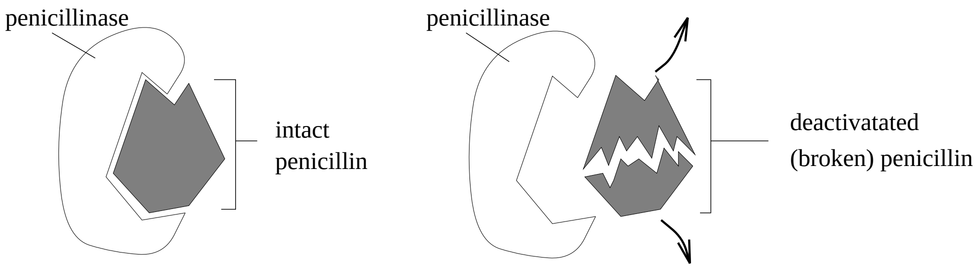
20. A particular collection of hypothalamic brain cells seem to be linked to appetite. These cells contain two different proteins, called Y and M. Here is a graph of the abundance of these two proteins in a typical hypothalamic cell as a function of time. What does this data tell us? Suppose that the key to weight loss is preventing high levels Y protein in these cells. You have been asked to design a drug for weight control. As a first guess, what should be the function of the drug?



21. A chemical *enzyme* is a molecule that "locks onto" a specific molecule called a *substrate*. The enzyme helps convert the substrate into some new molecule; i.e. the enzyme helps carry out a reaction involving the substrate. The *turnover rate* for an enzyme is defined to be the number of substrate molecules converted per second. Below is a table of optimal turnover numbers for some common biological enzymes. We start with a vessel containing water and add 10^8 substrate molecules and 100 enzyme molecules. In the third column of the table, find a linear function for the number of substrate molecules converted after t seconds. In the fourth column of the table, find a linear function for the number of substrate molecules remaining after t seconds.

enzyme	turnover rate	substrate converted	substrate remaining
carbonic anhydrase	600,000		
penicillinase	2000		
DNA polymerase I	15		
lysozyme	0.5		

22. A biochemist is investigating the enzyme *penicillinase* which will "deactivate" penicillin by breaking it into two pieces:



   (a) A vessel contains a solution of water, penicillinase and penicillin. Experimental calculations show that the rate at which penicillinase deactivates penicillin molecules is 2000 molecules/second; this is called the *turnover rate*. Find a linear function which computes the number of deactivated penicillin molecules after t seconds. Sketch the graph for the first 100 seconds. How many penicillin molecules are deactivated after 10 seconds? How long does it take to deactivate 50,000 penicillin molecules?

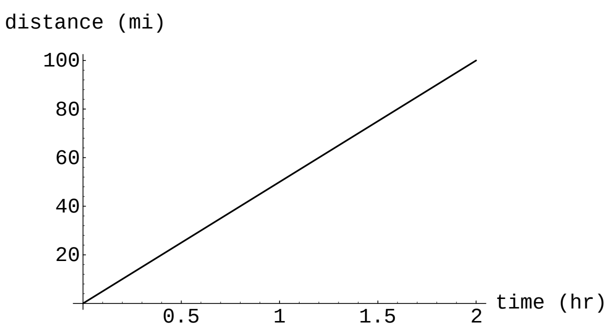
   (b) Adding an *inhibitor* to the solution will slow down the reaction. Theoretical calculations show that if α grams of inhibitor are added to the solution, then the new turnover rate is $\frac{2000}{1+\alpha^2}$ molecules/second. Find a linear function which computes the number of deactivated penicillin molecules after t seconds. (Your equation will involve the variable "t" AND the constant "α"; we usually call such an α a *parameter*.) For the first 100 seconds, in a common coordinate system, sketch the graphs of these functions for these α: $\alpha = 0, 1, 2, 3, 4$. Make sure to label each of your graphs with it's corresponding function.

   (c) Suppose the biochemist wants the solution to exhibit a turnover rate of 600 molecules/second. How many grams of inhibitor should be added?

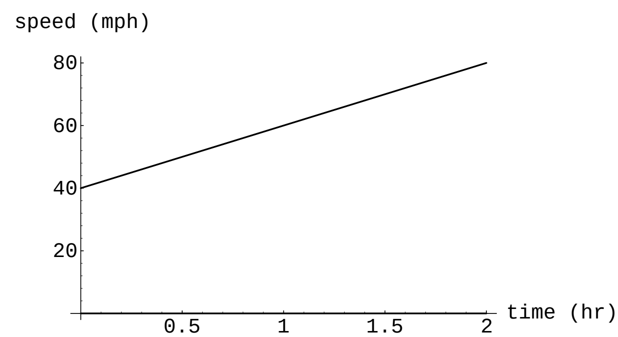
   (d) Suppose the biochemist wants exactly 6000 molecules of penicillin to be deactivated in a 25 second time period. How many grams of inhibitor should be added?

23. Two trucks (a Ford and a Chevy) pass through an intersection at the same time. The lefthand graph below relates time (in hours) and distance (in miles) for the Ford truck, beginning the instant it passes through the intersection. The righthand graph below relates time (in

hours) and speed (in mph) for the Chevy truck, beginning the instant it passes through the intersection. In both cases, these quantities are related by linear functions; what are the rules for these functions? What does the slope of each line tells us? Which vehicle is going faster at time $t = 15$ minutes?



FORD TRUCK

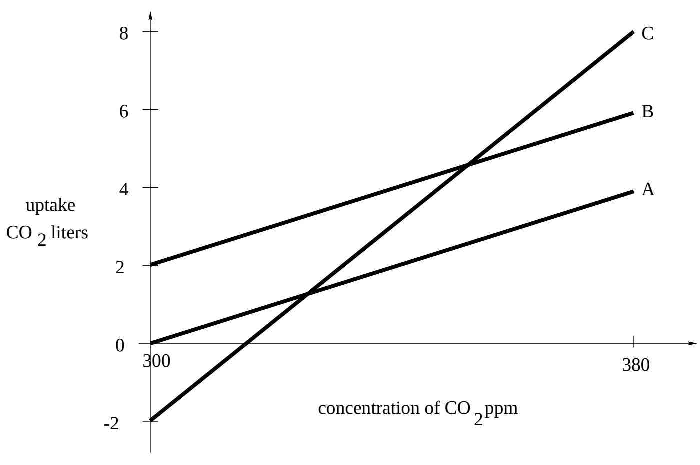


CHEVY TRUCK

24. *Photosynthesis* is a process carried out by plants that converts carbon dioxide (CO_2) and light energy into sugar and oxygen (O_2):

CO_2 + light energy $\Rightarrow$ sugar + O_2.

Suppose that a seed company scientist is studying three varieties of corn; call them A,B and C. For each variety, the scientist has carried out experimental measurements relating the CO_2 concentration of the air and the CO_2 uptake of the plant in one hour. Here is the experimental data; the units "ppm" mean "parts per million". In other words, if the CO_2 concentration of the air is 300 ppm, that means that out of 10^6 molecules of air, 300 will be CO_2.



(a) For each of the three lines, find a linear function whose graph is the line on an appropriate domain. Also describe the range in each case.

(b) If CO_2 concentration is 350 ppm, find the CO_2 uptake.

(c) If CO_2 concentration is 300 ppm, explain in words what each plant is doing.

(d) Under what conditions will each variety uptake 6 liters of CO_2?

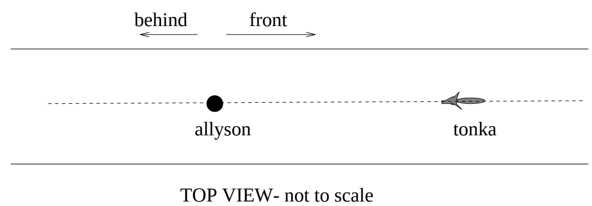
(e) Under what conditions will varieties A and B have the same CO_2 uptake?

(f) Under what conditions will varieties A and C have the same CO_2 uptake?

(g) Under what conditions will varieties B and C have the same CO_2 uptake?

(h) Typical CO_2 concentration levels in Seattle are around 340 ppm. Suppose you wanted Seattle to have conditions so that the uptake of variety C exceeds the uptake of variety A by 5 liters. By what percentage must the CO_2 concentration level increase?

25. Allyson is playing with her dog *Tonka* who starts 20 feet in front of her and runs back and forth along a straight line, as pictured. Lee has gathered data on Tonka's distance in front of Allyson at time t seconds; note that a negative value for this distance means that Tonka is behind Allyson.



Tonka's location (feet in front of Allyson)											
t	feet	t	feet	t	feet	t	feet	t	feet	t	feet
0	20	2	-2.5	4	-10	6	-2.5	8	20	10	57.5
1	6.9	3	-8.1	5	-8.1	7	6.9	9	36.9		

(a) Plot this data in a time vs. distance coordinate system.

(b) During each of the 10 one-second time intervals, compute the slope of a line connecting the two data points for that time interval. For example, during the first one-second time interval, the line would connect the data points (0,20) and (1,6.9).

(c) The slope calculation in b. is called the *average velocity* (in units of feet/sec) during the given time interval. When is the average velocity positive and negative in b?

(d) Connect your successive data points with straight lines and use this to estimate when Tonka is located right next to Allyson.