A Galois Correspondence in Batyrev's Construction of Mirror Pairs
Abstract

Toric variety is a branch of algebraic geometry where combinatorics and algebraic geometry substantiate each other. In this paper we will explore how to construct abstract toric variety that can be described purely by algebraic data without embedding into affine or projective spaces from concrete combinatorial data. Two method will be introduced. The first one uses a polytope and the second one a fan. We will present how to construct a fan associated with a polytope and prove that these two methods yield isomorphic toric varieties. Finally, we give an application to mirror symmetry. Given a finite morphism of reflexive pairs \((M_1, \Delta_1), (M_2, \Delta_2)\) of equal dimension, we obtain two families \(L(\Delta_i)\) of hypersurfaces in the projective toric varieties \(X_{\Delta_i}\) constructed from the polytopes \(\Delta_i\). \(L(\Delta_1)\) can then be viewed as a quotient of \(L(\Delta_2)\) by a finite abelian group \(A\). Mirror symmetry appears when we consider everything we have done in duality. The dual finite morphism provides another finite morphism between \((N_2, \Delta^*_2)\) and \((N_1, \Delta^*_1)\). It turns out the family \(L(\Delta^*_2)\) is a quotient of \(L(\Delta^*_1)\) by the dual finite abelian group \(A^* = \text{Hom}_\mathbb{Z}(A, \mathbb{C}^*)\). Moreover, the assignment \(L(\Delta) \mapsto L(\Delta^*)\) is a Galois correspondence between families of hypersurfaces constructed from reflexive polytopes.
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1 Acknowledgements

The first person that I want to thank for, purely from the view point of constructing this paper, for different considerations yield different persons to whom I am indebted, is without wonder my advisor: associate professor Jyh Haur Teh. A little more than a year a go, he accepted to guide me in the boundless and bottomless disciplines of mathematics. He was willing to discuss difficulties I encountered during the preparation of my thesis and spent a lot of time choosing subjects that meet both my interests and ability. Fortunately, he is not too tender to have spoiled me. Had it not been for his insistence on my establishing a Galois correspondence that resembles field’s version, I could not have understood what Batyrev’s Galois correspondence stands for. The other persons, by no means secondary, whom my thesis and I owed to are my parents, my relatives and my friends here in Tsing Hua. Perhaps they did not directly advise me on my thesis, their influences were indispensable. They provided a mind altering environment where I could get away from my work for some time. Interaction with them stimulated me to go forward. Talks with them exposed me to knowledge that is nowhere to be found in textbooks nor in papers.
2 Preliminaries

We will begin with some elementary notions used in this paper.

Definition 2.1. A lattice $M$ is a free abelian group of finite rank. The dual lattice of $M$ is the free abelian group $N := \text{Hom}_\mathbb{Z}(M, \mathbb{Z})$. There is a natural pairing $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$ given by $\langle a, \alpha \rangle \mapsto \alpha(a)$.

We may associate the dual lattices $M$ and $N$ with dual vector spaces: $M \otimes \mathbb{Z} \mathbb{R}$, $N \otimes \mathbb{Z} \mathbb{R}$ and $M \otimes \mathbb{Z} \mathbb{Q}$, $N \otimes \mathbb{Z} \mathbb{Q}$. We will write, for brevity, $M \otimes \mathbb{Z} \mathbb{R}$ (resp. $M \otimes \mathbb{Z} \mathbb{Q}$) and $N \otimes \mathbb{Z} \mathbb{R}$ (resp. $N \otimes \mathbb{Z} \mathbb{Q}$) respectively.

Definition 2.2. Let $S$ be a set in $M_{\mathbb{R}}$. The convex hull of $S$ is the smallest convex set in $M_{\mathbb{R}}$ that contains $S$. We denote the convex hull of $S$ by $\text{Conv}(S) = \{ \sum_{k=1}^{k} \lambda_i s_i | k \in \mathbb{N}, \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1, s_i \in S \}$.

Definition 2.3. A polytope in $M_{\mathbb{R}}$ is a convex hull of a finite set in $M_{\mathbb{R}}$.

Definition 2.4. Let $S$ be a set in $M_{\mathbb{R}}$. The affine hull is the smallest affine space in $M_{\mathbb{R}}$ that contains $S$. We will denote the affine hull of $S$ by $\text{aff}(S)$. Equivalently, $\text{aff}(S) = \{ \sum_{k=1}^{k} \lambda_i s_i | k \in \mathbb{N}, \sum_{i=1}^{k} \lambda_i = 1, s_i \in S \}$.

Given $u \neq 0$ in $N_{\mathbb{R}}$ and $b \in \mathbb{R}$, we get the hyperplane $H_{u,b} := \{ m \in M_{\mathbb{R}} | \langle m, u \rangle = b \} \subseteq M_{\mathbb{R}}$ and the closed half-space $H_{u,b}^+ := \{ m \in M_{\mathbb{R}} | \langle m, u \rangle \geq b \} \subseteq M_{\mathbb{R}}$.

Definition 2.5. A polyhedron $P$ is an intersection of finitely many closed half-spaces. The dimension of $P$ is defined to be the dimension of its affine hull: $\text{aff}(P)$.

We quote an useful fact proved in [6] and [5].

Proposition 2.6. $P$ is a polytope if and only if $P$ is a bounded polyhedron.

Definition 2.7. For any convex set $C$ in $M_{\mathbb{R}}$, we say $H_{u,b}$ is a supporting hyperplane of $C$ if $C \subseteq H_{u,b}^+$ and $C \cap H_{u,b} \neq \phi$. We say $F$ is a face of $C$ if there is a supporting hyperplane $H_{u,b}$ of $C$ such that $F = C \cap H_{u,b}$. In particular, we name $F$ as a facet of $C$ when $\dim F = \dim C - 1$ and a vertex of $C$ when $F$ is a 0 dimensional face of $C$.

We were informed that every polytope is a finite intersection of closed-half spaces. When the polytope $P$ is full dimensional, that is, $\dim P = \dim M_{\mathbb{R}}$ where $P \subseteq M_{\mathbb{R}}$, there is an elegant description of $P$ due to the following theorem:

Theorem 2.8. Let $P \subseteq M_{\mathbb{R}}$ be a polyhedron such that $\dim P = \dim M_{\mathbb{R}}$. Then the half-spaces $H_{u_1,b_1}^+, \cdots, H_{u_n,b_n}^+$ in an irredundant representation $P = H_{u_1,b_1}^+ \cap \cdots \cap H_{u_n,b_n}^+$ are uniquely determined. In fact, $F_i = P \cap H_{u_i,b_i}^+, i = 1, \cdots, n$ are the facets of $P$. 

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Therefore, we adopt the notation from [2] and write, if necessary, a polytope P as its facet representation:

\[ P = \bigcap_{F \text{ facet}} H_{u_F, -a_F}^+ = \{ m \in M_\mathbb{R} \mid \langle m, u_F \rangle \geq -a_F \text{ for all facets } F \prec P \}. \]

Moreover, if \( H_{u_F', -a_F'}^+ \) is another representation, then \((u_F, -a_F)\) and \((u_F', -a_F')\) differs by a multiple of positive scalar.

**Definition 2.9.** A lattice polytope is a polytope whose vertices are in \( M \subseteq M_\mathbb{R} \). A rational polytope is a polytope whose vertices are in \( M_\mathbb{Q} \subseteq M_\mathbb{R} \).

Now we introduce a special kind of polytope which is the core objects in mirror symmetry. If \( \Delta \) represents a lattice polytope in \( M_\mathbb{R} \) of dimension \( d \). We denote the set of vertices and the interior of \( \Delta \) by \( V(\Delta) \) and \( \text{int}(\Delta) \) respectively.

**Definition 2.10.** Assume \( 0 \in \text{int}(\Delta) \). The dual(polar) polytope of \( \Delta \) is

\[ \Delta^* := \{ y \in N_\mathbb{R} \mid \langle x, y \rangle \geq -1 \ \forall x \in \Delta \}. \]

\( \Delta \) is said to be reflexive if \( \Delta^* \) is also a lattice polytope.

We note without proof that \( \Delta^* \) is a rational polytope and that \( 0 \in \text{int}(\Delta^*) \). In this case, \( 0 \) is the its only interior lattice point. The proof of the fact that \( \Delta^* \) is \( d \)-dimensional can be found in [5]. Here is an useful remark. But we need some definitions first.

**Definition 2.11.** A lattice point \( u \in M_\mathbb{R} \) (resp. \( v \in N_\mathbb{R} \)) is primitive if \( u_k \) (resp. \( v_k \)) \( / \in M \) (resp. \( N \)) for all positive integer \( k \). A point \( x \in M_\mathbb{R} \) is said to have integral distance \( \delta \) from a facet \( F \) of \( \Delta \) if there exists a primitive lattice point \( u \in N_\mathbb{R} \) such that \( \langle u, F \rangle \equiv c \) for some \( c \in \mathbb{R} \) and \( \langle u, F \rangle - \langle u, x \rangle = \delta \).

**Remark 2.12.** A lattice polytope \( \Delta \) is reflexive iff every facet of \( \Delta \) has integral distance from its unique interior lattice point 0.

To verify the above remark, we will need the following terminology and theorem from [5] with some slight modifications.

**Definition 2.13.** Let \( W \) be a set in \( M_\mathbb{R} \) such that \( 0 \notin \text{aff}(W) \). The polarity \( \pi \) assigns \( W \) to another affine space

\[ \pi(W) = \cap_{w \in W} H_w := \{ x \mid \langle x, w \rangle = -1 \} \subseteq N_\mathbb{R} \]

of dimension \( d-1 \)-dim \( W \).

Note that the requirement \( 0 \notin \text{aff}(W) \) is essential because if \( \text{aff}(W) \) is linear then \( H_w \cap H_{2w} \) is empty for some nonzero \( w \in W \). Moreover this requirement will be implemented in the proof of the above dimension formula.

**Theorem 2.14.** Let \( P \) be a \( d \)-dimensional polytope in \( M_\mathbb{R} \), \( 0 \in P \), and let \( P^* \) be the polar polytope. Then
(1) If $F$ is a proper face of $P$, then $F^* := P^* \cap \pi(\text{aff}(F))$ is a proper face of $P^*$ and $\dim F^* = d - 1 - \dim F$.

(2) The assignment $F \mapsto F^*$ is an inclusion reversing and bijective map between the proper faces of $P$ and those of $P^*$.

**proof of remark:**

$(\Rightarrow)$ Let $F$ be a facet of $\Delta$. By theorem, $F^*$ is a vertex of $\Delta^*$ and by the dimension formula $\pi(\text{aff}(F))$ is a point. The dimension formula is applicable because $0 \in \text{int}(\Delta)$. So $0 \not\in \text{aff}(F)$. Thus we may let $\mathbb{R}^d \ni u = F^* = \pi(\text{aff}(F))$. Then by definition of $\pi$ we have $\langle u, F \rangle \equiv -1$. It remains to prove that $u$ is primitive and $-u$ will serve our needs. Since $\Delta$ is reflexive, $u \in \mathbb{N}_\mathbb{R}$. Write $F = \text{Conv}\{m_1, m_2, \ldots, m_s\}$ where $\{m_1, m_2, \ldots, m_s\} \subseteq \text{V}(\Delta) \subseteq \mathbb{R}^d$. Since $\langle u, F \rangle \equiv -1$, $\langle u, m_1 \rangle = -1$. Suppose $u^k$ is primitive for some positive integer $k$, then $\langle u^k, m_1 \rangle = -1$. Only if $k = 1$. Therefore, $u$ is primitive.

$(\Leftarrow)$ Let $F$ be a facet of $\Delta$. There is a primitive lattice $u$ such that $\langle u, F \rangle = 1$. So $-u \in \pi(\text{aff}(F)) = F^*$. Since $V(\Delta^*)$ arises from the set of all facets of $\Delta$ via the assignment $F \mapsto F^*$, we obtain $V(\Delta^*) \subseteq \mathbb{N}$.

It is noteworthy that one of the consequences of this proof is a necessary condition for $\Delta$ to be reflexive.

**Corollary 2.15.** Let $\Delta \subseteq \mathbb{R}^d$ be a $d$-dimensional lattice polytope. If $\Delta$ is reflexive, then $V(\Delta^*)$ consists of primitive lattices. Moreover, by the formula $(\Delta^*)^* = \Delta$ one can go further to conclude that $V(\Delta)$ consists of primitive lattice points is also a necessary condition for reflexivity of $\Delta$.

To shorten many statements in section 5 and 6, we adopt the following definition.

**Definition 2.16.** Let $\Delta_i \subseteq (M_i)_{\mathbb{Z}}$, $i=1,2$ be two reflexive polytopes of equal dimension.

We say $(\Delta_i, M_i)$ forms a reflexive pair. A homomorphism $\phi : M_1 \rightarrow M_2$ is a finite morphism of reflexive pairs if

1. $\phi(\Delta_1) = \Delta_2$
2. faces of $\Delta_1$ is mapped bijectively onto those of $\Delta_2$.

**Proposition 2.17.** If $\phi : (M_1, \Delta_1) \rightarrow (M_2, \Delta_2)$ is a finite morphism of reflexive pairs, then $\phi^*(N_2, \Delta_2^*) \rightarrow (N_1, \Delta_1^*)$ is again a finite morphism of reflexive pairs.

**Proof.** If $y \in \Delta_2^*$, $y \in (\phi(F_1))^*$ if and only if $\phi^*(y) \in F_1^*$ where $F_1 \prec \Delta_1$ and $F_1^* \prec \Delta_1^*$. Therefore,

$$\phi(F_1) = F_2 \iff (\phi(F_1))^* = F_2^* \iff \phi^*(F_2^*) = F_1^*.$$

Next we need the other two terminologies: Cones and fans.

**Definition 2.18.** Let $S$ be a set in $\mathbb{R}^d$. The cone generated by $S$ is

$$\text{Cone}(S) := \{\sum_{i=1}^n \lambda_i s_i | \lambda_i \geq 0, s_i \in S, n \in \mathbb{N}\}.$$

If $S$ is finite, then $\text{Cone}(S)$ is said to be a polyhedral cone.
Definition 2.19. For a polyhedral cone \( \sigma \) Define \( \sigma^\vee := \{ y \in N_\mathbb{R} : \langle x, y \rangle \geq 0 \} \). \( \sigma^\vee \) is called the dual cone of \( \sigma \).

Definition 2.20. Let \( \sigma \subseteq M_\mathbb{R} \) be a polyhedral cone. If the generating set of \( \sigma \) can be chosen to be lattices in \( M \), then we name \( \sigma \) as rational polyhedral cone.

Definition 2.21. A cone \( \sigma \) is said to be strongly convex if \( \emptyset \) is a vertex of \( \sigma \).

Definition 2.22. A fan \( \Sigma \) in \( N_\mathbb{R} \) consists of a finite collection of strongly convex rational polyhedral cones in \( N_\mathbb{R} \) satisfying:

- If \( \sigma \in \Sigma \), then every face of \( \sigma \) is also in \( \Sigma \).
- If \( \sigma, \tau \in \Sigma \), then \( \sigma \cap \Sigma \) is a face of each.

Definition 2.23. The set \( |\Sigma| = \cup_{\sigma \in \Sigma} \sigma \subseteq N_\mathbb{R} \) is the support of \( \Sigma \). The fan is complete if \( |\Sigma| = N_\mathbb{R} \). For each \( d \), \( \Sigma(d) \) denotes the set of \( d \)-dimensional cones of \( \Sigma \).
3 Affine Toric Varieties

Definition 3.1. Let the group structure of \((C^*)^n = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{C} \setminus \{0\}\}\) be given by component wise multiplication for all integer \(n\). A torus \(T\) is an affine variety isomorphic to \((C^*)^s\) for some integer \(s\). Moreover, \(T\) inherits a group structure from \((C^*)^s\) via the isomorphism.

Definition 3.2. A character of a torus \(T \simeq (C^*)^n\) is a morphism \(\chi^m : T \rightarrow \mathbb{C}^*\) that is also a group homomorphism. The set \(M\) of all characters of a torus \(T\) has a natural group structure: \((\chi^{m_1} \cdot \chi^{m_2})(t) := \chi^{m_1}(t) \cdot \chi^{m_2}(t)\) for all \(t \in T\).

Remark 3.3. \(m = (a_1, \ldots, a_n) \in \mathbb{Z}^n\) gives a character \(\chi^m : (C^*)^n \rightarrow \mathbb{C}^*\) defined by \(\chi^m(t_1, \ldots, t_n) = t_1^{a_1} \cdots t_n^{a_n}\).

From theory of linear algebraic group, all characters of \((C^*)^n\) arise in this way. Let \(\varphi : \mathbb{C}^* \rightarrow T\) be the isomorphism, then the assignment \(\chi^m \mapsto \chi^m \circ \varphi\) gives a group isomorphism between \(M\) and \(\mathbb{Z}^n\). Thus the characters of \((C^*)^n\) form a free abelian group of rank \(n\), the dimension of \(T\).

Definition 3.4. A one parameter subgroup of \(T \simeq (C^*)^n\) is a morphism \(\lambda^u : C^* \rightarrow T\) that is also a group homomorphism. The set \(N\) of all one-parameter subgroups of a torus \(T\) has a natural group structure: \((\lambda^{u_1} \cdot \lambda^{u_2})(c) := \lambda^{u_1}(c) \cdot \lambda^{u_2}(c)\) for all \(c \in C^*\).

Remark 3.5. \(u = (b_1, \ldots, b_n) \in \mathbb{Z}^n\) gives a one-parameter subgroup \(\lambda^u : C^* \rightarrow (C^*)^n\) defined by \(\lambda^u(t) = (t^{b_1}, \ldots, t^{b_n})\).

All one parameter subgroup arises in this way. It follows that the group of one-parameter subgroup is isomorphic to \(\mathbb{Z}^n\). If the group of characters of \(T\) is \(M\) and \(N\) is the group of one-parameter subgroup, we will denote \(T\) as \(T_N\). The reason, in concern for the exposition of main subjects, will be postponed.

Definition 3.6. An affine toric variety is an irreducible affine variety \(V\) containing a torus \(T_N \simeq (C^*)^n\) as a Zariski open subset such that the action of \(T_N\) on itself extends to an algebraic action on \(V\). That is, the action \((t, v) \mapsto t \cdot v\) is given by a morphism \(T_N \times V \rightarrow V\).

Remark 3.7. For every \(t \in T_N\), the map \(\phi_t : V \rightarrow V\) defined by \(\phi_t(v) = t \cdot v\) is a morphism.

Proof. Simply observe that \(\phi_t\) is the composition of morphisms:

\[ V \xrightarrow{\phi_{t,v}} T_N \times V \xrightarrow{t \cdot v} V \]

\(\square\)

Example 3.8. We claim that the curve \(C = V(x^3 - y^2) \subseteq \mathbb{C}^2\) is an affine toric variety. Since \(x^2 - y^2\) is irreducible, \(C\) is an irreducible affine variety.

\[ C \setminus \{0\} = C \cap (C^*)^2 = \{(t^2, t^3) \mid t \in C^*\} \simeq C^* \]
is a torus which is open. The isomorphism is \( t \mapsto (t^2, t^3) \). Therefore, the group structure of \( C \setminus \{0\} \) can be calculated via \( (x, y) \ast (x', y') = \text{preimage of } \frac{y'}{y} = (xx', yy') \), which is just the componentwise multiplication. Hence the action of \( C \setminus \{0\} \) on itself extends to an algebraic action on \( C \).

Our first attempt to construct toric varieties from combinatorial data arises from affine semigroup \( S \subseteq M \) where \( M \) is a lattice.

**Definition 3.9.** An affine semigroup is a set \( S \) with an associative binary operation and an identity element such that

1. The binary operation is commutative
2. \( S \) is finitely generated, meaning that there is a finite set \( \mathcal{A} \subseteq S \) such that

\[
\mathbb{N}\mathcal{A} = \left\{ \sum_{m \in \mathcal{A}} a_m m \mid a_m \in \mathbb{N} \right\}.
\]

**Definition 3.10.** Given an affine semigroup \( S \subseteq M \), the \( \mathbb{C} \)-algebra \( \mathbb{C}[S] \) is the vector space with basis \( S \). Writing

\[
\mathbb{C}[S] = \left\{ \sum_{m \in S} c_m \chi^m \mid c_m \in \mathbb{C} \right\}
\]

where \( c_m = 0 \) for all but finitely many \( m \), the multiplication in \( \mathbb{C}[S] \) is induced by

\[
\chi^m \cdot \chi^{m'} = \chi^{m+m'}.
\]

In the proposition that follows, \( \text{Spec}(R) \) means the collection of all maximal ideals of the ring \( R \).

**Proposition 3.11.** Let \( S \subseteq M \) be an affine semigroup. Then

1. \( \mathbb{C}[S] \) is an integral domain and finitely generated \( \mathbb{C} \)-algebra.
2. \( \text{Spec}(\mathbb{C}[S]) \) is an affine toric variety.

**Example 3.12.** Given a lattice \( M \), \( \text{Spec}(\mathbb{C}[M]) \) is an affine toric variety. If \( \{m_1, \cdots, m_n\} \) is a basis for \( M \), then the generating set for \( M \) as an affine semigroup is just \( \{\pm m_1, \cdots, \pm m_n\} \). In fact, it is a torus with character lattice \( M \).

When we have a rational polyhedral cone \( \sigma \subseteq \mathbb{N}_\mathbb{R} \), we may obtain an affine semigroup via Gordan’s Lemma and an affine toric variety:

**Lemma 3.13.** The lattice points

\[
S_{\sigma} = \sigma^\vee \cap M \subseteq M
\]

form an affine semigroup.

**Proposition 3.14.** Let \( \sigma \subseteq \mathbb{N}_\mathbb{R} \) be a rational polyhedral cone with affine semigroup \( S_{\sigma} = \sigma^\vee \cap M \). Then

\[
U_{\sigma} = \text{Spec}(\mathbb{C}[S_{\sigma}]) = \text{Spec}(\mathbb{C}[\sigma^\vee \cap M])
\]

is an affine toric variety. Furthermore, \( \dim U_{\sigma} = n \Leftrightarrow T_N \) is a torus of \( U_{\sigma} \Leftrightarrow \sigma \) is strongly convex.
4 Projective Toric Varieties

**Definition 4.1.** A projective toric variety is an irreducible projective variety $V$ that contains a torus $T^*_N \simeq (C^*)^n$ as a Zariski open subset such that the action of $T^*_N$ on itself extends to an algebraic action of $T^*_N$ on $V$. (By algebraic action, we mean an action $T^*_N \times V \rightarrow V$ given by a regular map.)

Let us check that the projective space $\mathbb{P}^n$ is a projective toric variety.

**Proposition 4.2.** The projective variety $\mathbb{P}^n$ is a projective toric variety whose torus has character lattice

$$M_n = \{ (c_0, \ldots, c_n) \in \mathbb{Z}^{n+1} \mid \sum_{i=0}^n c_i = 0 \}.$$ 

and the one parameter subgroup

$$N_n = \mathbb{Z}^{n+1}/\mathbb{Z}(1, \ldots, 1).$$

**Proof.** First, note that $\mathbb{P}^n$ is irreducible since it determines the zero ideal in $\mathbb{C}[x_0, \ldots, x_n]$. Second, the open set $T^*_n = \mathbb{P}^n \backslash V(x_0 \cdots x_n) = \{ (a_0, \ldots, a_n) \in \mathbb{P}^n \mid a_0 \cdots a_n \neq 0 \} = \{ (1, t_1, \ldots, t_n) \in \mathbb{P}^n \mid t_1, \ldots, t_n \in \mathbb{C}^* \} \simeq (\mathbb{C}^*)^n$. From the isomorphism the action on $T^*_n$ is given by

$$\begin{align*}
[a_0 : \ldots : a_n][b_0 : \ldots : b_n] &= [1 : \frac{a_1}{a_0} : \ldots : \frac{a_n}{a_0}][b_0 : \ldots : b_n] \\
&= [1 : \frac{a_1 b_0}{a_0} : \ldots : \frac{a_n b_n}{a_0}] = [a_0 b_0 : \ldots : a_n b_n].
\end{align*}$$

Therefore, the action of $T^*_n$ on itself extends to component wise multiplication on $\mathbb{P}^n$.

$p$ is a projective toric variety with torus $T^*_n$. We also compute the character lattice $M_n$. The following exact sequence is useful:

$$1 \longrightarrow \mathbb{C}^* \longrightarrow (\mathbb{C}^*)^{n+1} \overset{\pi}{\longrightarrow} \mathbb{P}^n \longrightarrow 1$$

where $\pi$ maps $(a_0, \ldots, a_n)$ to $[a_0 : \ldots : a_n]$. Since $\pi$ is surjective, the assignment $m \mapsto m \circ \pi$ for $m \in M_n$ is an injective map to the group of characters of $(\mathbb{C}^*)^{n+1}$. Thus computing $M_n$ is equivalent to computing the image of the above assignment. We claim that

$$M_n = \{ (c_0, \ldots, c_n) \in \mathbb{Z}^{n+1} \mid \sum_{i=0}^n c_i = 0 \}.$$ 

Given $m \in M_n$,

$$m \circ \pi((a_0, \ldots, a_n)) = m([a_0 : \ldots : a_n]) = m([1 : \frac{a_1}{a_0} : \ldots : \frac{a_n}{a_1}]) = (\frac{a_1}{a_0})^{m_1} \cdots (\frac{a_n}{a_0})^{m_n} = a_0^{-m_1} \cdots a_n^{-m_n} a_1^{m_1} \cdots a_n^{m_n}$$

for some $(m_1, \ldots, m_n) \in \mathbb{Z}^n$. But what $m \circ \pi$ actually gives us is the $(n+1)$ tuple: $(-m_1 \cdots -m_n, m_1, \ldots, m_n)$ whose components has sum equal to 0. Conversely, let $(c_0, \ldots, c_n) \in \mathbb{Z}^{n+1}$ with $\sum_{i=0}^n c_i = 0$. Define $m$ by $m([x_0 : \ldots : x_n]) = x_0^{c_0} \cdots x_n^{c_n}$. This is a homogenous polynomial of degree 0 and is regular on $T^*_n$. Then $m \in M_n$ and $m \circ \pi = (c_0, \ldots, c_n)$. □
Following affine semigroups, polytopes are our second combinatorial objects to construct toric varieties. In general, we construct a projective toric variety from a given finite set \( \mathcal{A} \) of lattices. To be more precise, let \( M \) be a lattice and \( \mathcal{A} = \{m_1, \ldots, m_s\} \subseteq M \), then we have the map

\[
\Phi_{\mathcal{A}} : T_N \longrightarrow \mathbb{C}^s, \ t \mapsto (\chi^{m_1}(t), \ldots, \chi^{m_s}(t)).
\]

Denote \( \pi \) to be the projection map of \((\mathbb{C}^*)^s\) onto

\[
T_{\mathbb{P}^{s-1}} := \{[a_0, \ldots, a_{s-1}] \in \mathbb{P}^{s-1} | a_0 \cdots a_{s-1} \neq 0\}.
\]

**Definition 4.3.** Given a finite set \( \mathcal{A} \subseteq M \), the projective variety \( X_{\mathcal{A}} \) is the Zariski closure of \((\pi \circ \Phi)(T_N)\) in \( \mathbb{P}^{s-1} \).

**Proposition 4.4.** \( X_{\mathcal{A}} \) is a toric variety where

1. \((\pi \circ \Phi)(T_N)\) is a torus in \( X_{\mathcal{A}} \) and \((\pi \circ \Phi)(T_N) = X_{\mathcal{A}} \cap T_{\mathbb{P}^{s-1}}\).
2. the algebraic action of the torus on \( X_{\mathcal{A}} \) is given by componentwise multiplication.

**Definition 4.5.** An affine semigroup \( S \subseteq M \) is saturated if for all \( k \in \mathbb{N} \setminus \{0\} \) and \( m \in M \), \( km \in S \) implies \( m \in S \).

**Definition 4.6.** A lattice polytope \( P \subseteq M \) is very ample if for every vertex \( m \in P \), the semigroup generated by \( P \cap M - m = \{m - m' | m' \in P \cap M\} \) is saturated in \( M \).

**Proposition 4.7.** If \( P \subseteq M \) is a full dimensional lattice polytope, then there exists a \( k \in \mathbb{N} \) such that \( kP \) is very ample.

**Definition 4.8.** Let \( P \subseteq M \) be a full dimensional lattice polytope. Then we define the toric variety of \( P \) to be \( X_{kP \cap M} \) where \( k \) is any positive integer such that \( kP \) is very ample.

**Remark 4.9.** There might be two integers \( k \) and \( l \) such that \( kP \) and \( lP \) are very ample. We will see that the above definition is well defined once we know the construction of a fan from a polytope is invariant under dilation and translation of the polytope.

**Proposition 4.10.** Given \( \mathcal{A} = \{m_1, \ldots, m_s\} \subseteq M \), let \( P = \text{Conv}(\mathcal{A}) \subseteq M \) and set \( J = \{j \in \{1, \ldots, s\} | m_j \text{ is a vertex of } P\} \). Then

\[
X_{\mathcal{A}} = \bigcup_{j \in J} X_{\mathcal{A}} \cap U_j
\]

This means it suffices to consider vertex of a polytope \( P \) to describe all affine pieces of \( X_P \). To make the affine pieces of \( X_P \) more concrete, we now construct a fan from a polytope.

Let \( P \subseteq M \) be a full dimensional lattice polytope. Write \( P \) as its facet representation

\[
P = \{m \in M | \langle m, u_F \rangle \geq -a_F \text{ for all } F\}
\]

Let \( Q \preceq P \) be a face of \( P \). We associate each \( Q \) a cone

\[
\sigma_Q = \text{Cone}(u_F | F \text{ contains } Q) \subseteq N_R.
\]
Proposition 4.11. The collection of cones \( \Sigma_P = \{ \sigma_Q \mid Q \subseteq P \} \) is a fan.

Proposition 4.12. Let \( X_{P \cap M} \) be the projective toric variety of the very ample polytope \( P \subseteq M_\mathbb{R} \). Assume further that \( P \) is full dimensional, say, \( n \). Then:

- For each vertex \( v \in P \cap M \), the affine piece \( X_{P \cap M} \cap U_v = \text{Spec}(\mathbb{C}[\text{Cone}(P \cap M - v)]) \).
- Writing \( \sigma_v = \text{Cone}(P \cap M - v)^\vee \), we have \( \dim \sigma_v = n \) and \( \sigma_v \) is strongly convex rational polyhedral. That is, \( X_{P \cap M} \cap U_v \) is the affine \( n \)-dimensional toric variety \( U_{\sigma_v} = \text{Spec}(\mathbb{C}[\sigma_v^\vee \cap M]) \).
- \( X_{P \cap M} \) has \( T_N \) as its torus whose character lattice is \( M \).
- If \( v \neq w \) are two vertices of \( P \) and \( Q \) is the smallest face of \( P \) containing \( v \) and \( w \), then \( X_{P \cap M} \cap U_v \cap U_w = U_{\sigma_Q} = \text{Spec}(\mathbb{C}[\sigma_Q^\vee \cap M]) \).
- \( U_{\sigma_v} \supseteq (U_{\sigma_v})_{\lambda^{v-w}} = U_{\sigma_Q} = (U_{\sigma_w})_{\lambda^{w-v}} \subseteq U_{\sigma_w} \).

5 Construction of Toric Variety From a Fan

In general, suppose we have a finite collection \( \{ V_\alpha \} \) of affine varieties and for all pairs \( \alpha, \beta \), we have Zariski open sets \( V_{\beta \alpha} \subseteq V_\alpha \) and isomorphisms \( g_{\beta \alpha} : V_{\beta \alpha} \simeq V_{\alpha \beta} \) satisfying the following conditions

1. \( V_{\alpha \alpha} = V_\alpha \)
2. \( g_{\alpha \alpha} = 1_{V_\alpha} \).
3. \( g_{\alpha \beta} = g_{\beta \alpha}^{-1} \).
4. \( g_{\gamma \alpha} = g_{\gamma \beta} \circ g_{\beta \alpha} \) on \( V_{\beta \alpha} \cap V_{\gamma \alpha} \) for all \( \alpha, \beta, \gamma \).

The abstract variety \( X \) glued from \( \{ V_\alpha \} \) is defined by

\[
\prod_\alpha V_\alpha \overset{\sim}{\longrightarrow} X
\]

where \( a \sim b \) if and only if \( a \in V_\alpha, b \in V_\beta \) for some \( \alpha, \beta \) with \( b = g_{\beta \alpha}(a) \).

Let \( \pi : \prod_\alpha V_\alpha \longrightarrow X \) be the projection and \( \varphi_\beta : V_\beta \longrightarrow \prod_\alpha V_\alpha \) be the canonical injection. Then we define the Zariski topology on \( X \) by \( V \subseteq X \) is open if and only if \( \varphi_\beta^{-1}(\pi^{-1}(V)) \) is open in \( V_\beta \) for every \( \beta \).

Remark 5.1. If \( X_1 \) is a variety glued from \( \{ V_\alpha \} \), \( X_2 \) from \( \{ U_\beta \} \). Then the product \( X_1 \times X_2 \) is a variety glued from \( \{ V_\alpha \times U_\beta \} \). In fact, if \( V_{\gamma \alpha} U_{\delta \beta} \) are open subsets of \( V_\alpha U_\beta \) and \( g_{\gamma \alpha}, h_{\delta \beta} \) are the isomorphisms, then for every \( V_\alpha \times U_\beta \), there are open subsets \( V_{\gamma \alpha} \times U_{\delta \beta} \) and isomorphisms \( g_{\gamma \alpha} \times h_{\delta \beta} : V_{\gamma \alpha} \times U_{\delta \beta} \longrightarrow V_{\alpha \gamma} \times U_{\beta \delta} \) that satisfy gluing condition.

Definition 5.2. A variety \( X \) is separated if the image of the diagonal map \( \Delta : X \longrightarrow X \times X \) defined by \( \Delta(x) = (x, x) \) is Zariski closed in \( X \times X \).

We have a sufficient condition for a variety to be separated.

Lemma 5.3. Let \( X \) be a variety obtained by gluing the affine varieties \( \{ V_\alpha \} \) along open subsets \( V_{\alpha \beta} \subseteq V_\beta \) by isomorphisms \( g_{\alpha \beta} : V_{\alpha \beta} \simeq V_{\beta \alpha} \). Then \( X \) is separated if the image of \( \Lambda : V_{\alpha \beta} \longrightarrow V_\alpha \times V_\beta \) defined by \( \Lambda(p) = (p, g_{\alpha \beta}(p)) \) is Zariski closed for all \( \alpha, \beta \).
Proof. We will describe the set \( \varphi_{g\gamma}^{-1}\pi^{-1}(\Delta(X)) \) where \( \varphi_{g\gamma} : V_\delta \times V_\gamma \to \coprod_{\alpha\beta} V_\alpha \times V_\beta \) is the canonical injection and \( \pi \) is the projection map.

\[
(z_1, z_2) \in \varphi_{g\gamma}^{-1}\pi^{-1}(\Delta(X)) \iff (z_1, z_2) \in V_\delta \times V_\gamma \text{ and } [z_1, z_2] = [x, x] \text{ for some } x \in V_\alpha
\]

\[
\iff g_{\alpha\delta} \times g_{\alpha\gamma}(z_1, z_2) = (g_{\alpha\delta}(z_1), g_{\alpha\gamma}(z_2)) = (x, x).
\]

But \( g_{\alpha\delta} = g_{\alpha\gamma} \circ g_{\gamma\delta} \) so \( g_{\alpha\delta}(z_1) = g_{\alpha\gamma}g_{\gamma\delta}(z_1) = g_{\alpha\gamma}(z_2) \) if and only if \( z_2 = g_{\gamma\delta}(z_1) \). Hence the image of \( \Lambda \) is \( \varphi_{g\gamma}^{-1}\pi^{-1}(\Delta(X)) \) from which the lemma follows. \( \square \)

We generalize the definition of toric variety that includes abstract varieties.

**Definition 5.4.** A toric variety is an irreducible variety \( X \) containing a torus \( T_N \simeq (\mathbb{C}^*)^n \) as a Zariski open subset such that the action of \( T_N \) on itself extends to an algebraic action of \( T_N \) on \( X \).

**Proposition 5.5.** Let \( X \) be the abstract variety glued from the collection \( T \) of affine varieties \( \{V_\alpha\} \). Then \( X \) is irreducible if and only if each \( V_\alpha \in T \) is irreducible.

*Proof.* Suppose \( X \) is irreducible and where \( U, W \subset V_\alpha \) are two nonempty open subsets of \( V_\alpha \). It is sufficient to prove that \( U \cap W \neq \emptyset \). But first let us show that \( \pi_\alpha(U) \) is open in \( X \). If \( z \in \varphi_{g\gamma}^{-1}\pi^{-1}(\pi_\alpha(U)) \), then \([z] = [x] \) for some \( x \in V \). By definition of the equivalence relation \( \sim \), \( g_{\alpha\gamma}(z) = x \). Therefore \( z \in g_{\alpha\gamma}^{-1}(U) \). It is clear that \( g_{\alpha\gamma}^{-1}(U) \subseteq \varphi_{g\gamma}^{-1}\pi^{-1}(\pi_\alpha(U)) \).

As a result, \( \varphi_{g\gamma}^{-1}\pi^{-1}(\pi_\alpha(U)) = g_{\alpha\gamma}^{-1}(U) \), which is open since \( g_{\alpha\gamma} \) is an isomorphism and \( U \cap V_{\gamma\alpha} \) is open. However, \( \gamma \) is arbitrary so \( \pi_\alpha(U) \) is open in \( X \). Similarly, \( \pi_\alpha(W) \) is open in \( X \). Since \( X \) is irreducible, \( \pi_\alpha(U) \cap \pi_\alpha(W) \neq \emptyset \). By the fact that \( \pi \) is injective on \( \pi_\alpha(V_\alpha) \) and \( \varphi_\alpha \) is injective:

\[
\pi_\alpha(U) \cap \pi_\alpha(W) = \pi_\alpha(U \cap W) \neq \emptyset.
\]

So \( U \cap W \neq \emptyset \). Conversely, suppose each \( V_\alpha \in T \) is irreducible, let \( U \) and \( W \) be two nonempty open sets in \( X \). Then \( \varphi_{g\gamma}^{-1}\pi^{-1}(U) \) and \( \varphi_{g\gamma}^{-1}\pi^{-1}(W) \) are two nonempty open sets in \( V_\gamma \) for each \( \gamma \). Consequently, \( \varphi_{g\gamma}^{-1}\pi^{-1}(U) \cap \varphi_{g\gamma}^{-1}\pi^{-1}(W) \neq \emptyset \). Hence there is a \( x \in V_\gamma \) such that \([x] \in U \cap W \). \( \square \)

Now suppose \( \Sigma \subseteq (N_\mathbb{R}) \) is a fan. Each cone \( \sigma \in \Sigma \) gives an affine toric variety \( U_{\sigma} = \text{Spec}(\mathbb{C}[\sigma^\vee \cap M]) \). We have the following description of \( U_{\sigma_1 \cap \sigma_2} \).

**Proposition 5.6.** \( U_{\sigma_1 \cap \sigma_2} = U_{\sigma_1} \cap U_{\sigma_2} \). Moreover, \( U_{\sigma_1 \cap \sigma_2} \) is an open subset of \( U_{\sigma_i} \) for \( i=1, 2 \).

Therefore, the isomorphism on \( U_{\sigma_i} \cap U_{\sigma_j} \) is just the identity. So the glueing condition is satisfied easily.

**Proposition 5.7.** Let \( \Sigma \) be a fan in \( N_\mathbb{R} \). The variety \( X_\Sigma \) is a separated toric variety.

*Proof.* The variety \( X_\Sigma \) is irreducible since each \( U_{\sigma} \) is. The torus \( T_N = \text{Spec}(\mathbb{C}[M]) = \text{Spec}(\mathbb{C}[0^\vee \cap M]) \) is contained in each \( U_{\sigma} \) where \( 0 \) is the trivial cone in \( \Sigma \). Although every \( U_{\sigma} \) is made disjoint from each other, the equivalence relation pertaining to isomorphisms, which are identities, identifies each \( T_N \) as one torus. That \( T_N \) is open in \( X \) follows from \( T_N \) being open in each \( U_{\sigma} \). As for algebraic action, for every \( \sigma_i \in \Sigma \), there is a morphism
$\psi_i : T_N \times U_{\sigma_i} \to U_{\sigma_i}$ that extends the action of $T_N$ on itself. $\psi_i$ and $\psi_j$ agree on $T_N \times U_{\sigma_i} \cap U_{\sigma_j}$ and can be glued to a morphism $\psi : T_N \times X_\Sigma \to X_\Sigma$ that extends the algebraic action of $T_N$ on itself.

To see that $X_\Sigma$ is separated, it is sufficient to show that for each pair of cones $\sigma_1$ and $\sigma_2$, the image of the diagonal map $\Delta : U_{\sigma_1 \cap \sigma_2} \to U_{\sigma_1} \times U_{\sigma_2}$ is Zaraski closed. But $\Delta$ comes from the $C$-algebra homomorphism

$$\Delta^* : C[S_{\sigma_1}] \otimes_C C[S_{\sigma_2}] \to C[S_T]$$

defined by $\chi^m \otimes \chi^n \mapsto \chi^{m+n}$. By proposition 3.1.3 in [2], $\Delta^*$ is surjective, so that the image of $\Delta$ is a Zaraski closed in $U_{\sigma_1} \times U_{\sigma_2}$.

We will see how to identify $X_{\Sigma_P}$ and $X_P$. First, three lemmas

**Lemma 5.8.** If $f, g : Y \to X$ are morphisms, then $Z = \{ y \in Y \mid f(y) = g(y) \}$ is Zaraski closed in $Y$.

**Proof.** This follows from if $F : Y \to X$ is given by $F(y) = (f(y), g(y))$, then $Z = F^{-1}(\Delta(X))$.

**Lemma 5.9.** Let $P \subseteq M$ be a full dimensional lattice polytope and $\Sigma_P \subseteq N$ the normal fan constructed from $P$. Let $\Sigma'$ be the subcollection consisting of $\sigma_v$ where $v \in V(P)$. Then $X_{\Sigma'} \cong X_{\Sigma_P}$.

**Proof.** Note that $X_{\Sigma'}$ is still an abstract variety glued from the affine pieces $\{U_{\sigma_v}\}$ where $v \in V(P)$. It is a separated toric variety. The proof of this fact is the same as that in the above proposition. For any cone $\sigma_Q \subseteq \Sigma_P$, where $Q$ is a face of $P$. Let $v \in V(Q) \subseteq V(P)$. Then $\sigma_Q \cap \sigma_v = \sigma_Q$ by proposition 2.3.7 in [2]. Therefore, $\sigma_Q \subseteq \sigma_v$. So the identity map on $N$ is compatible with $\Sigma_P$ and $\Sigma'$. Conversely, since $\Sigma'$ is a subcollection of $\Sigma_P$, the identity map is trivially compatible with $\Sigma_P$ and $\Sigma'$. By theorem 3.3.4 in [2], there are toric morphisms $\varphi : X_{\Sigma'} \to X_{\Sigma_P}$, $\psi : X_{\Sigma_P} \to X_{\Sigma'}$ such that $\varphi \circ \psi = \psi \circ \varphi = 1$ on $T_N$. Since $T_N$ is open in $X_{\Sigma'}$ and $X_{\Sigma_P}$, by proposition 3.0.18 and theorem 3.1.5 in [2], $\varphi \circ \psi = 1_{X_{\Sigma_P}}$, $\psi \circ \varphi = 1_{X_{\Sigma'}}$.

**Lemma 5.10.** Let $P$ be a full dimensional lattice polytope in $M$. Then, there exist $k \in \mathbb{R}$ such that $kP$ is very ample. Furthermore, the normal fan $\Sigma_P$ is invariant under translation and dilation. Namely, $\Sigma_{kP+l} = \Sigma_P$ for all $k \in \mathbb{R}$ and $l \in M$.

**Proof.** omitted

**Proposition 5.11.** Let $P$ be a lattice polytope and $\Sigma_P$ the fan constructed from $P$. Then $X_P \cong X_{\Sigma_P}$.

**Proof.** One first chooses $k \in \mathbb{R}$ such that $kP = P'$ is very ample. By previous proposition, the varieties $X_P \cap U_v$ where $v \in V(P')$ are affine pieces of $X_P$. $X_P$ is actually the result of glueing these affine pieces. Therefore, $X_{\Sigma_{P'}} \cong X_P$. But $X_{\Sigma_P} = X_{\Sigma_{P'}}$, so $X_{\Sigma_P} \cong X_P$ now follows.
6 Group Action on Toric Varieties

The major goal here is to prove the following: Given two reflexive pairs $(M_1, \Delta_1)$, $(M_2, \Delta_2)$ of equal dimension and a homomorphism of lattices $\phi : M_1 \rightarrow M_2$ where each face of $\Delta_1$ is mapped bijectively onto the face of $\Delta_2$, there is a relation:

$$P_{\Delta_2,M_2}/G \simeq P_{\Delta_1,M_1}.$$ 

where $P_{\Delta_i,M_i}$ is the projective toric variety associated with $\Delta_i$, $G$ is the finite abelian group $N_1/\phi^*(N_2)$ and $\phi^* : N_2 \rightarrow N_1$ is the dual map of $\phi$. Since $X_{\Sigma_{\Delta_1}, N_1} \simeq P_{\Delta_1, M_1}$, we will prove this in the frame of fan's construction of toric varieties. To transform this problem into the language of fans, note that $\Delta_1$ is reflexive, so each cone $\sigma_i \in \Sigma_{\Delta_1}$ is generated by the exactly one face of $\Delta_i$. Moreover, it can be shown that

1. each face of $\Delta_i^*$ is mapped bijectively onto $\Delta_i^*$.
2. $\phi(\Delta_1) = \Delta_2$ (resp. $\phi^*(\Delta_2^*) = \Delta_1^*$) implies $\phi$(resp. $\phi^*$) is injective.

Therefore, we formulate the main task as: Given a homomorphism $\phi : M_1 \rightarrow M_2$ of lattices of equal dimension, two fans $\Sigma_2 \in (N_2)_R$, $\Sigma_1 \in (N_1)_R$ where for each $\sigma_1 \in \Sigma_1$, there is one and only one $\sigma_2 \in \Sigma_2$ such that $\phi^*(\sigma_2) = \sigma_1$, there is a relation

$$X_{\Sigma_2, N_2}/G \simeq X_{\Sigma_1, N_1}.$$ 

and focus on lattice homomorphisms that is injective whose dual is also injective. To prove this, we need many auxiliary lemmas. In the lemma that follows, we use the intrinsic nature of the pairing between a lattice and its dual: Let $M$ be the lattice of characters of a torus $T$ and $N$ be its lattice of one parameter subgroups. Given a character $\chi^m$ and a one parameter subgroup $\lambda^n$, the composition $\chi^m \circ \lambda^n : C^* \rightarrow C^*$ is a character (one parameter subgroup) of the torus $C^*$. Since all character of $C^*$ is of the form $u \mapsto u^k$ where $u \in C^*$ and $k \in \mathbb{Z}$, define the paring $\langle \cdot, \cdot \rangle : M \times N \rightarrow \mathbb{Z}$ by $(\chi^m \lambda^n)(u) = u^{(m,n)}$ where $u \in C^*$. Then the paring identifies $N$ with $\text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ and vice versa.

**Lemma 6.1.** There is a canonical isomorphism $N \otimes_{\mathbb{Z}} C^*$ given by $u \otimes t \mapsto \lambda^u(t)$. Thus it is customary to write $T$ as $T_N$.

**Proof.** Fix an isomorphism $\eta : T_N \simeq (C^*)^s$ where $s =$rank $N$. Define the bilinear map $\theta : N \times C^* \rightarrow T_N$ by $\theta(u, t) = \lambda^u(t)$. The universal property allows us to infer the map $\overline{\theta} : N \otimes_{\mathbb{Z}} C^* \rightarrow T_N$ where $\overline{\theta}(u \otimes t) = \lambda^u(t)$ is a well-defined group homomorphism. To find its inverse, denote $\{m_1, \cdots, m_s\}$ to be the basis of $M_1$ and $\{n_1, \cdots, n_s\}$ the basis of $N_1$ such that under the isomorphism $\eta$, $\chi^m$ corresponds to i-th component projection from $(C^*)^s$ onto $C^*$ and $n_j$ corresponds to j-th component injection from $C^*$ into $(C^*)^s$.

The conveniences of doing this are

1. $\langle m_i, n_j \rangle = \delta_{ij}$ so that for any $n \in N$,
   
   $$n = \sum_i (m_i, n)n_i.$$ 

2. $\prod_i \lambda^{n_i}(\chi^{m_i}(t)) = t$ for all $t \in T_N$. 

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Then define \( \vartheta : T_N \to N \otimes \mathbb{C}^* \) by \( t \mapsto \sum_i n_i \otimes \chi^{m_i}(t) \). A simple calculation shows that
\[
\vartheta \circ \theta(u \otimes t) = \vartheta(\lambda^u(t)) = \sum_i n_i \otimes \chi^{m_i}(\lambda^u(t))
\]
\[
= \sum_i n_i \otimes t^{(m_i,u)} = \sum_i (m_i,u)n_i \otimes t = u \otimes t
\]
and
\[
\bar{\vartheta} \circ \vartheta(t) = \bar{\vartheta}(\sum_i n_i \otimes \chi^{m_i}(t)) = \prod_i \bar{\vartheta}(n_i \otimes \chi^{m_i}(t))
\]
\[
= \prod_i \lambda^n(\chi^{m_i}(t)) = t.
\]

**Lemma 6.2.** There is an isomorphism \( T_N \to \text{Hom}_\mathbb{Z}(M, \mathbb{C}^*) \) given by \( t \mapsto \phi_t \) where \( \phi_t(m) = \chi^m(t) \).

**Proof.** Let \( \{m_1, \cdots, m_s\} \) and \( \{n_1, \cdots, n_s\} \) be basis of \( M \) and \( N \) as in the above lemma. The assignment \( t \mapsto \phi_t \) is clearly a group homomorphism. To see it is injective, suppose \( \chi^{m}(t) = 1 \in \mathbb{C}^* \) for all \( m \in M \). Then \( \chi^{n_i}(t) = 1 \in \mathbb{C}^* \). Since \( \lambda^{n_i} \) is a group homomorphism, we have \( T_N \ni 1 = \lambda^{n_i}(1) = \lambda^{n_i}(\chi^{m_i}(t)) = t^{(m_i,n_i)} = t \). Therefore, the assignment is injective. To see it is surjective, let \( \psi \in \text{Hom}_\mathbb{Z}(M, \mathbb{C}^*) \). Consider \( \prod_i \lambda^m(\psi(m_i)) \in T_N \).

Then for all \( m \in M \),
\[
\chi^{m}(\prod_i \psi(m_i)) = \prod_i \chi^{m}(\lambda^m(\psi(m_i)))
\]
\[
= \prod_i \psi(m_i)^{(m,m_i)} = \psi(\sum_i (m_i,n_i)n_i) = \psi(m).
\]

Combining the above two lemmas, we obtain an isomorphism
\( \nu : N \otimes \mathbb{C}^* \to \text{Hom}_\mathbb{Z}(M, \mathbb{C}^*) \)
defined by the composition:
\[
u(u \otimes t) \mapsto \lambda^u(t) \mapsto \phi_{\lambda^u(t)}.
\]
So \( \nu(u \otimes t)(m) = t^{(m,u)} \). This ensures the following relation:

**Lemma 6.3.** Let \( \phi : M_1 \to M_2 \) be a homomorphism of lattices. Denote the dual map by \( \phi^* : N_2 \to N_1 \). There is another dual map which we denote by \( \bar{\phi} : \text{Hom}_\mathbb{Z}(M_2, \mathbb{C}^*) \to \text{Hom}_\mathbb{Z}(M_1, \mathbb{C}^*) \). Finally, there is a map which we also denote by \( \phi^* : N_2 \otimes \mathbb{C}^* \to N_1 \otimes \mathbb{C}^* \) defined by \( \phi^* \otimes 1_{\mathbb{C}^*} \). Then there is the commutative diagram
\[
\begin{array}{ccc}
N_2 \otimes \mathbb{C}^* & \xrightarrow{\phi^*} & N_1 \otimes \mathbb{C}^* \\
\nu_2^{-1} & & \nu_1 \\
\text{Hom}_\mathbb{Z}(M_2, \mathbb{C}^*) & \xrightarrow{\bar{\phi}} & \text{Hom}_\mathbb{Z}(M_1, \mathbb{C}^*)
\end{array}
\]
Proof. Denote the paring between $M_i$ and $N_i$ by $\langle \ , \ \rangle$. Given $m_1 \in M_1$ and $\varrho \in Hom_\mathbb{Z}(M_2, \mathbb{C}^*)$. Suppose $\nu_2(u \otimes t) = \varrho$. Then
\[
(\overline{\phi}(\varrho))(m_1) = \psi(\phi(m_1)) = \nu_2(u \otimes t)(\phi(m_1)) = t(\phi(m_1),u) \cdot \varrho
\]
\[
= t(\phi(m_1),\varrho(u)) = \chi^\varrho(u)(\chi^{m_1}(t))
\]
\[
= \nu_1(\varrho(u) \otimes t)(m_1) = \nu_1(\varrho(\phi(m_1))(m_1) = (\nu_1 \varrho \nu_2^{-1})(\varrho)(m_1)
\]
\[
\square
\]

Lemma 6.4. Let $V = \text{Spec}(\mathbb{C}[\mathcal{S}])$ be an affine toric variety of the affine semigroup $\mathcal{S}$. Then there is a bijective correspondence between points $p \in V$ and semigroup homomorphisms $S \to \mathbb{C}$, where $\mathbb{C}$ is considered as a semigroup under multiplication. If $T_N$ is the torus of $V$, let $t \in T_N$ and $\gamma$ be the affine semigroup homomorphism for $p$. Then the action $t \cdot p$ corresponds to the semigroup homomorphism: $m \mapsto \chi^m \gamma(m)$.

Proof. See proposition 1.3.1 in [2] and the comments after it. \[
\square
\]

Proposition 6.5. Let $\phi : M_1 \to M_2$ be an injective homomorphism of lattices with equal dimension. Denote the dual map of $\phi$ by $\phi^\ast : N_2 \to N_1$. The map $\phi^\ast : (M_1)_R \to (M_2)_R$ stands for the map obtained by $\phi \otimes 1_R$. Assume further that $\phi^\ast$ is injective. Denote $\phi^\ast_{T_N} : N_2 \otimes \mathbb{C}^* \to N_1 \otimes \mathbb{C}^*$ to be the map $\phi^\ast \otimes 1_{\mathbb{C}^*}$. Let $\sigma_2 \subseteq (N_2)_R$ and $\phi^\ast(\sigma_2) \subseteq \phi^\ast(N_2)_R$ be two strongly convex rational polyhedral cones. Then

1. There are isomorphisms
\[
G = N_1/\phi^\ast(N_2) \cong Hom_\mathbb{Z}(M_2/\phi(M_1), \mathbb{C}^*) \cong \ker(T_{N_2} \to T_{N_1})
\]
such that $g \in G$ acts on $U_{\sigma_2,N_2}$.

2. $G$ also acts on $\mathbb{C}[\phi^\ast(\sigma_2)^\vee \cap M_2]$ with ring of invariants
\[
\mathbb{C}[\phi^\ast(\sigma_2)^\vee \cap M_2]^G = \mathbb{C}[\phi^\ast(\sigma_2)^\vee \cap \phi(M_1)].
\]
and the morphism $\pi : \text{Spec}(\mathbb{C}[\sigma_2^\vee \cap M_2]) \to \text{Spec}(\mathbb{C}[\sigma_2^\vee \cap \phi(M_1)])$ induced by the inclusion $\mathbb{C}[\sigma_2^\vee \cap \phi(M_1)] \hookrightarrow \mathbb{C}[\sigma_2^\vee \cap M_2]$ is constant on $G$-orbits and induces a bijection
\[
U_{\sigma_2,N_2}/G \cong U_{\sigma_2,\phi(M_1)^*}.
\]

3. There is an isomorphism
\[
\psi : U_{\sigma_2,\phi^\ast(M_1)} \to U_{\phi^\ast(\sigma_2),N_1}
\]
where $\psi^\ast : \mathbb{C}[\phi^\ast(\sigma_2)^\vee \cap M_1] \to \mathbb{C}[\sigma_2^\vee \cap \phi(M_1)]$ is given by $\chi^{m_1} \to \chi^{\phi^\ast(m_1)}$ for $m_1 \in \phi^\ast(\sigma_2)^\vee \cap M_1$.

4. The toric morphism $\varphi : U_{\phi^\ast(\sigma_2),M_2} \to U_{\phi^\ast(\sigma_2),\phi(M_1)}$ extended from the morphism $\phi^\ast : T_{N_2} \to T_{N_1}$ is constant on $G$-orbits.

5. The diagram commutes
\[
\begin{array}{ccc}
U_{\sigma_2,N_2} & \xrightarrow{\varphi} & U_{\phi^\ast(\sigma_2),N_1} \\
\downarrow{\pi} & & \downarrow{\psi} \\
U_{\sigma_2,\phi(M_1)^*} & & \\
\end{array}
\]
6. $\varphi$ induces a bijection

\[ U_{\varphi^*(\sigma_2), M_2}/G \cong U_{\varphi^*(\sigma_2), \phi(M_1)}. \]

**Proof.** We have the short exact sequence

\[ 0 \longrightarrow M_1 \xrightarrow{\varphi} M_2 \longrightarrow M_2/\phi(M_1) \longrightarrow 0. \]

Using $T_N = \text{Hom}_\mathbb{Z}(M, \mathbb{C}^*)$ and applying $\text{Hom}_\mathbb{Z}(M_2/\phi(M_1), \mathbb{C}^*)$, we claim that the sequence

\[ 1 \longrightarrow \text{Hom}_\mathbb{Z}(M_1/\phi(M_1), \mathbb{C}^*) \longrightarrow T_{N_2} \xrightarrow{\overline{\varphi}} T_{N_1} \longrightarrow 1. \]

is exact. Because $\text{Hom}_\mathbb{Z}(-, \mathbb{C}^*)$ is left exact, it remains to prove $\overline{\varphi}$ is surjective. By previous lemma, it is equivalent to check that $\phi_T^*: N_2 \otimes \mathbb{C}^* \rightarrow N_1 \otimes \mathbb{C}^*$ is surjective. Let $\{n_2^1, \ldots, n_2^n\}$ be a basis of $N_2$ so $\{\phi^*(n_2^1), \ldots, \phi^*(n_2^n)\}$ is a basis of $\phi^*(N_2)$. Given $n_1 \otimes t \in N_1 \otimes \mathbb{C}^*$, since $(N_1)_Q = (\phi^*(N_2))_Q$, write $n_1 = \sum q_i \phi^*(n_2^i)$ for some $q_i \in \mathbb{Q}$. Let $l$ be a common multiple of the denominators of $q_i$. Then

\[ n_1 \otimes t = \sum q_i \phi^*(n_2^i) \otimes t = \sum q_i \phi^*(\lambda_i) \otimes s = \phi^*(\sum q_i n_2^i) \otimes s \]

where $s^i = t$. Therefore, $n_1 \otimes t \in \phi^*(N_2)$.

We have the inclusions

\[ \phi^*(N_2) \subseteq N_1 \subseteq (N_1)_Q \text{ and } \phi(M_1) \subseteq M_2 \subseteq \phi(M_1)_Q \cong (M_1)_Q. \]

The pairing between $M_1$ and $N_1$ induces a pairing $\phi(M_1) \times N_1 \longrightarrow \mathbb{Z}$ defined by $\langle \phi(m_1), n_1 \rangle = \langle m_1, n_1 \rangle$ which again induces a pairing $\phi(M_1)_Q \times (N_1)_Q \rightarrow \mathbb{Q}$. Define the map

\[ M_2/\phi(M_1) \times N_1/\phi^*(N_2) \longrightarrow \mathbb{C}^* \]

by

\[ ([m_2], [n_1]) \mapsto e^{2\pi i \langle m_2, n_1 \rangle}. \]

We claim that it is well defined and induces $G = N_1/\phi^*(N_2) \cong \text{Hom}_\mathbb{Z}(M_2/\phi(M_1), \mathbb{C}^*)$. If $[m_2] = [m'_2]$ and $[n_1] = [n'_1]$. Then $m_2 = m'_2 + \phi(m_1)$ for some $m_i \in M_1$ and $n_1 = n'_1 + \phi^*(n_2)$ for some $n_2 \in N_2$. Therefore,

\[ \langle m_2, n_1 \rangle - \langle m'_2, n'_1 \rangle = \langle \phi(m_1), n_1 \rangle + \langle \phi(m_1), \phi^*(n_2) \rangle + \langle m'_2, \phi^*(n_2) \rangle \in \mathbb{Z}. \]

Whence $e^{2\pi i \langle m_2, n_1 \rangle} = e^{2\pi i \langle m'_2, n'_1 \rangle}$. Define $\nu: G \longrightarrow \text{Hom}_\mathbb{Z}(M_2/\phi(M_1), \mathbb{C}^*)$ by

\[ \nu([n_1])([m_2]) = e^{2\pi i \langle m_2, n_1 \rangle}. \]

If $\psi([n_1])([m_2]) = 1 \in \mathbb{C}^*$ for all $[m_2] \in M_2/\phi(M_1)$. Then $\langle m_2, n_1 \rangle \in \mathbb{Z}$ for all $m_2 \in M_2$. Then $n_1 \in M_2^* = \phi^*(N_2)$. Therefore, $\nu$ is injective. Given $\psi \in \text{Hom}_\mathbb{Z}(M_2/\phi(M_1), \mathbb{C}^*)$, let $\{m_2^1, \ldots, m_2^n\}$ be a basis for $M_2$ such that there are $d_1, \ldots, d_n \in \mathbb{Z}$ and $\{d_1 m_2^1, \ldots, d_n m_2^n\}$ is a basis for $\phi(M_1)$. Then $\{[m_2^1], \ldots, [m_2^n]\}$ generates $M_2/\phi(M_1)$. Since $M_2/\phi(M_1) \cong \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_n}$, $M_2/\phi(M_1)$ is finite. Hence $\psi([m_2^n]) = e^{2\pi i q_j}$ for some $q_j \in \mathbb{Q}$. It suffices to find $n_1 \in N_1$ such that $\langle n_1, v_j \rangle = q_j$. Theories concerning solving system of linear
equations guarantee such \( n_1 \in (N_1)_\mathbb{Q} \) exists. But \( d_j m_j^2 \in \phi(M_1) \) implies \( \psi([d_j m_j^2]) = 1 \) which in turn gives \( (d_j m_j^2, n_1) \in \mathbb{Z} \). Whence \( \langle \phi(M_1), n_1 \rangle \in \mathbb{Z} \) so \( n_1 \in N_1 \). Then \( v(n_1) = \psi \). Whence \( v \) is surjective.

Therefore, \( G \) may be viewed as a subgroup of \( T_{N_2} \). Doing so allows us to say the group \( G \) acts on \( U_{\sigma_2, N_2} \). The action of \( G \) on the coordinate ring is given by

\[
g \cdot \chi^{m_2} = g([m_2])^{-1}\chi^{m_2}, \quad m_2 \in \phi^*(\sigma_2)^\vee \cap M_2.
\]

Now \( g[m_2] = 1 \) for all \( g \in G \) if and only if \( e^{2\pi i \langle m_2, n_1 \rangle} = 1 \) for all \( n_1 \in N_1 \). This is equivalent to \( m_2 \in \phi(M_1)^* \) since under the paring \( \langle \cdot, \cdot \rangle : \phi(M_1)_\mathbb{Q} \times (N_1)_\mathbb{Q} \rightarrow \mathbb{Q} \) the dual of \( N_1 \) is \( \phi(M_1) \). Therefore, the ring of invariants

\[
\mathbb{C}[\phi^*(\sigma_2)^\vee \cap M_2]^G = \{ f \in \mathbb{C}[\phi^*(\sigma_2)^\vee \cap M_2] \mid g \cdot f = f \text{ for all } g \in G \},
\]

is precisely \( \mathbb{C}[\phi^*(\sigma_2)^\vee \cap \phi(M_1)] \). It was said in proposition 1.3.18 in [2] that when a group \( G \) acts on an affine variety \( \text{Spec}(R) \) where \( R \) and \( R^G \) are finitely generated \( \mathbb{C} \)-algebra without nilpotent, the morphism \( \pi : \text{Spec}(R) \rightarrow \text{Spec}(R^G) \) induced by the inclusion \( R^G \hookrightarrow R \) is constant on \( G \)-orbit and induces a bijection \( V/G \simeq \text{Spec}(R^G) \).

Note that if \( m_1 \in M_1, \langle m_1, \phi^*(\sigma_2) \rangle_1 \geq 0 \) if and only if \( \langle \phi(m_1), \sigma_2 \rangle_2 \geq 0 \). Hence the assignment

\[
\phi^*(\sigma_2)^\vee \cap M_2 \rightarrow \sigma_2^\vee \cap \phi(M_1)
\]

given by \( m_1 \mapsto \phi(m_1) \)

provides an isomorphism between affine semigroups. Therefore, the assignment \( \chi^{m_1} \mapsto \chi^{\phi(m_1)} \) gives an isomorphism of \( \mathbb{C} \)-algebras.

By proposition 1.3.14 and 1.3.15 in [2], the map \( \varphi : U_{\sigma_2, N_2} \rightarrow U_{\phi^*(\sigma_2), N_1} \) is toric and satisfies

\[
\varphi(g \cdot x) = \varphi(g) \cdot \varphi(x)
\]

for all \( g \in G \) and \( x \in X \). To prove that \( \varphi(g \cdot x) = \varphi(x) \), it would be much easier to proceed from combinatorial data. That is, the semigroup homomorphisms of corresponding to \( \varphi(g \cdot x) \) and \( \varphi(x) \) coincide. By lemma, if \( \gamma : \phi^*(\sigma)^\vee \cap M_1 \rightarrow \mathbb{C} \) is the semigroup homomorphism for \( \varphi(x) \), then \( \varphi(g) \cdot \varphi(x) \) corresponds to the semigroup homomorphism \( \phi^*(\sigma)^\vee \cap M_1 \rightarrow \mathbb{C} \), given by \( m_1 \mapsto \chi^{m_1}(\varphi(g))\gamma(m_1) \). Since

\[
\chi^{m_1}(\varphi(g)) = \varphi^*(\chi^{m_1})(g) = \chi^{\phi(m_1)}(g).
\]

However, \( g \in N_1/\phi^*(N_2) \) satisfies

\[
\chi^{\phi(m_1)}(g) = e^{2\pi i \langle \phi(m_1), n_1 \rangle} = 1
\]

if \( g = [n_1] \). Therefore, the semigroup homomorphisms coincide so that

\[
\varphi(g \cdot x) = \varphi(x).
\]

That is, \( \varphi \) is constant on \( G \)-orbits. To say \( \varphi \) yields a bijection on \( G \)-orbits, we will need to verify

1. \( \varphi \) is surjective.
2. \( \varphi(x) = \varphi(y) \) if and only if \( x \) and \( y \) lie in the same \( G \)-orbit. \\
\( \varphi \) being surjective follows immediately from 5. So we prove 5. Again combinatorial data is much easier to deal with, so we prove the corresponding \( \mathbb{C} \)-algebra homomorphism \( \pi^* \circ \psi^* = \varphi^* \). Given \( \chi^{m_1} \in \mathbb{C}[\phi^*(\sigma_2) \cap M_1] \), \( \varphi^*(\chi^{m_1}) = \chi^{\phi(m_1)} = \pi^* \circ \psi^*(\chi^{m_1}) \) since \( \pi^* \) is the inclusion \( \mathbb{C}[\sigma_2^* \cap \phi(M_1)] \hookrightarrow \mathbb{C}[\sigma_2^* \cap M_2] \). Therefore,
\[
\varphi = \pi \circ \psi,
\]
which is a composition of surjective maps so \( \varphi \) is surjective. Finally, if \( \varphi(x) = \varphi(y) \), then \( \pi(x) = \pi(y) \). Since \( G \) is a finite reductive group and \( \pi \) induces bijection on \( G \)-orbits, proposition 5.0.8 and 5.0.9 in [2] implies \( x \) and \( y \) lie in the same orbit. \( \blacksquare \)

**Remark 6.6.** A frequently used fact in the above proposition is that under the pairing \( \langle \ , \ \rangle : \phi(M_1)_Q \times (N_1)_Q \longrightarrow Q, \phi(M_1) \) is the dual of \( N_1 \) and \( \phi^*(N_2) \) is the dual of \( M_2 \).

**Proof.** We consider the second statement only. The proof of the first should be entirely analogous. Let \( \{m_1^n, \cdots, m_i^n\} \) be a basis of \( M_1 \). Since \( \phi \) is injective, \( \{\phi(m_1^n), \cdots, \phi(m_i^n)\} \) is a basis of \( \phi(M_1) \). By theorem 1.6, Chapter II in [3], let \( \{x_1, \cdots, x_n\} \) be a basis of \( M_2 \) such that \( \{d_1 x_1, \cdots, d_n x_n\} \) is a basis of \( \phi(M_1) \). Denote the natural paring \( M_1 \times N_1 \longrightarrow \mathbb{Z} \) by \( \langle \ , \ \rangle, \ i = 1, 2 \). Given \( m_2 \in M_2 \subseteq \phi(M_1)_Q \) and \( \phi^*(n_2) \) where \( n_2 \in N_2 \), write
\[
m_2 = \sum_i l_i x_i - \sum_i \frac{l_i}{d_i} d_i x_i.
\]
Since \( d_i x_i \in \phi(M_1) \), we write
\[
d_i x_i = \sum_j k_{ij} \phi(m_2^j)
\]
and obtain
\[
\langle m_2, \phi^*(n_2) \rangle = \sum_i \sum_j \frac{l_i}{d_i} k_{ij} \langle \phi(m_2^j), \phi^*(n_2) \rangle
\]

\[
= \sum_i \sum_j \frac{l_i}{d_i} k_{ij} \langle m_2^j, \phi^*(n_2) \rangle_1 = \sum_i \sum_j \frac{l_i}{d_i} k_{ij} \langle \phi(m_2^j), n_2 \rangle_2
\]

\[
= \sum_i \frac{l_i}{d_i} \langle \sum_j k_{ij} \phi(m_2^j), n_2 \rangle_2 = \sum_i \frac{l_i}{d_i} \langle d_i x_i, n_2 \rangle_2
\]

\[
= \sum_i l_i \langle x_i, n_2 \rangle_2 \in \mathbb{Z}.
\]
Therefore, \( N_2 \subseteq M_2^* \). Conversely, let \( \{n_2^1, \cdots, n_2^n\} \) be a basis of \( N_2 \) so \( \{\phi^*(n_2^1), \cdots, \phi^*(n_2^n)\} \) is a basis of \( \phi^*(N_2) \). Given \( n_1 \in N_1 \), suppose \( \langle m_2, n_1 \rangle \in \mathbb{Z} \) for all \( m_2 \in M_2 \), then since \( (N_1)_Q = (\phi^*(N_2))_Q \), write \( n_1 = \sum_i q_i \phi^*(n_2^i) \) for some \( q_i \in \mathbb{Q} \). Now we have
\[
\sum_i q_i \langle m_2, \phi^*(n_2^i) \rangle = \sum_i q_i \langle m_2, n_2^i \rangle_2 \in \mathbb{Z} \forall m_2 \in M_2.
\]
Let \( \{x_1, \cdots, x_n\} \) be the basis of \( M_2 \) such that \( \langle x_i, n_2^j \rangle_2 = \delta_{ij} \). Then \( \langle x_i, n_1 \rangle = q_i \in \mathbb{Z} \). Therefore, \( n_1 \in \phi^*(N_2) \). \( \blacksquare \)
Here is our main result

**Theorem 6.7.** Let $\phi : M_1 \rightarrow M_2$ be an injective homomorphism of lattices of equal dimension, $N_i = \text{Hom}_{\mathbb{Z}}(M_i, \mathbb{Z})$, $\Sigma_i$ be fans in $(N_i)_{\mathbb{R}}$ and $\phi^*: N_2 \rightarrow N_1$ be the dual map of $\phi$. Denote $N_2/\phi^*(N_1)$ as $G$. There is an inclusion $\phi(M_1) \hookrightarrow M_2$ that gives rise to another inclusion $N_2 \hookrightarrow \phi(M_1)^*$. In brevity we denote $\phi(M_1)^*$ as $N_2'$. Assume further that $\phi^*$ is injective and each cone $\sigma_i \in \Sigma_1$ is $\phi^*(\sigma_2)$ for one and only one $\sigma_2 \in \Sigma_2$. Then

1. Viewing $\Sigma_2$ as a fan in $(N_2)_{\mathbb{R}}$, the affine varieties $\{U_{\sigma_i,N_2'}\}_{\sigma_2 \in \Sigma_2}$ glued together to yield an abstract toric variety $X_{\Sigma_2,N_2'}$.

2. $G$ acts on $X_{\Sigma_2,N_2}$. The morphism $\pi : X_{\Sigma_2,N_2} \rightarrow X_{\Sigma_2,N_2'}$ obtained by gluing each $\pi_{\sigma_i} : U_{\sigma_i,N_2} \rightarrow U_{\sigma_i,N_2'}$ where $\pi_{\sigma_i}$ induces $U_{\sigma_i,N_2}/G \simeq U_{\sigma_i,N_2'}$ as in previous proposition induces $X_{\Sigma_2,N_2}/G \simeq X_{\Sigma_2,N_2'}$.

3. The toric morphism $\varphi : X_{\Sigma_2,N_2} \rightarrow X_{\Sigma_1,N_1}$ is constant on $G$-orbits and induces a bijection on $G$-orbits $X_{\Sigma_2,N_2}/G$ and points in $X_{\Sigma_1,N_1}$.

**Proof.** Part 1 follows from construction of an abstract toric variety from a fan. For part 2, $G$ acting on $X_{\Sigma_2,N_2}$ follows from identifying $G$ as a subgroup of $T_{N_2}$. We need to prove $\{\pi_{\sigma_i}\}_{\sigma_2}$ satisfy gluing condition. Once more combinatorial data proves their supreme value. Write $\pi_{\sigma_i} |_{U_{\sigma_i \cap \sigma_j,N_2}}$ as the composition of

$$U_{\sigma_i \cap \sigma_j,N_2} \rightarrow U_{\sigma_i,N_2} \rightarrow U_{\sigma_i,N_2'}.$$

Write $\pi_{\sigma_i \cap \sigma_j}$ as the composition of

$$U_{\sigma_i \cap \sigma_j,N_2} \rightarrow U_{\sigma_i,N_2 \cap \sigma_j,N_2} \rightarrow U_{\sigma_i,N_2'}.$$

Then it is easy to see that $(\pi |_{U_{\sigma_i \cap \sigma_j,N_2}})$ and $(\pi_{\sigma_i \cap \sigma_j})$ are inclusions $\mathbb{C}[\sigma_i^\vee \cap \phi(M_1)] \hookrightarrow \mathbb{C}[\sigma_i \cap \sigma_j^\vee \cap M_2]$. Whence $\pi_{\sigma_i} |_{U_{\sigma_i \cap \sigma_j,N_2}} = \pi_{\sigma_i \cap \sigma_j}$. An analogous argument shows that $\pi_{\sigma_j} |_{U_{\sigma_i \cap \sigma_j,N_2}} = \pi_{\sigma_i \cap \sigma_j}$. Whence $\pi_{\sigma_i} |_{U_{\sigma_i \cap \sigma_j,N_2}} = \pi_{\sigma_j} |_{U_{\sigma_i \cap \sigma_j,N_2}}$ Denote the glued morphism by $\pi$. Since $G$ is a finite group, it is reductive. Proposition 5.0.9 in [2] ensures that each $\pi_{\sigma_i}$ is a good categorical quotient. Applying proposition 5.0.12, in [2], we see that $\pi$ is also a good categorical quotient. If we prove $\pi(x) = \pi(y)$ if and only if $x$ and $y$ lie in the same $G$-orbit, we may apply proposition 5.0.8 to get the bijection

$$X_{\Sigma_2,N_2}/G \simeq X_{\Sigma_2,N_2'}$$

induced by $\pi$. Suppose $\pi(x) = \pi(y)$ and $x \in U_{\sigma_i,N_2}$ and $y \in U_{\sigma_j,N_2}$. Then

$$\pi(x) \in U_{\sigma_i,N_2} \cap U_{\sigma_j,N_2'} = U_{\sigma_i \cap \sigma_j,N_2}.$$

Since $\pi_{\sigma_i \cap \sigma_j}$ is surjective, there is a $z \in U_{\sigma_i \cap \sigma_j,N_2}$ such that

$$\pi_{\sigma_i \cap \sigma_j}(z) = \pi_{\sigma_i}(z) = \pi(x) = \pi_{\sigma_j}(z) = \pi_{\sigma_j \cap \sigma_i}(z).$$

Therefore, there is a $g \in G$ and $h \in G$ such that $g \cdot z = x$ and $h \cdot z = y$. Hence $gh^{-1} \cdot y = x$ so $x$ and $y$ lie in the same $G$-orbit. The converse is easy because if $x \in U_{\sigma_i,N_2}$,

$$\pi(g \cdot x) = \pi_{\sigma_i}(g \cdot x) = \pi_{\sigma_i}(x) = \pi(x).$$
For part 3, \( \varphi' \)'s begin constant on \( G \) orbits follows at once from the fact that it is the glued morphism of \( \varphi_{\sigma_i} : U_{\sigma_i,N_2} \to U_{\varphi(\sigma_i),N_1} \). One may by arguments similar to part 2 to obtain \( \varphi(x) = \varphi(y) \) if and only if \( x \) and \( y \) lie in the same \( G \)-orbit. Moreover, we have the following commutative diagram

\[
\begin{array}{ccc}
X_{\Sigma_2,N_2} & \xrightarrow{\varphi} & X_{\Sigma_1,N_1} \\
\downarrow{\pi} & & \downarrow{\psi} \\
X_{\Sigma_2,N_2'} & & \\
\end{array}
\]

where \( \psi \) is the isomorphism obtained by gluing \( \psi_{\sigma_i} : U_{\sigma_i,N_2'} \to X_{\sigma_1,N_1} \) as in previous proposition. Therefore, \( \varphi \) is surjective and we obtained the desired relation

\[
X_{\Sigma_2,N_2}/G \simeq X_{\Sigma_1,N_1}.
\]

\[
\square
\]

## 7 Hypersurfaces in Toric Varieties

**Definition 7.1.** A Laurent polynomial \( f = f(X) \) is a finite linear combination of elements of the character lattice \( M \) of the torus \( T \)

\[
f(X) = \sum c_m X^m
\]

where \( c_m \in \mathbb{C} \). For every Laurent polynomial \( f \), the convex hull of all \( m \) where \( c_m \neq 0 \) is called the Newton polyhedra \( \Delta(f) \) of \( f \). Every Laurent polynomial \( f \) with its Newton polyhedra \( \Delta \) defines the affine hypersurface

\[
Z_{f,\Delta} = \{ X \in T \mid f(X) = 0 \}.
\]

Let \( \overline{Z}_{f,\Delta} \) be the Zaraki closure of \( Z_{f,\Delta} \) in \( X_{\Delta} \). For any face \( \theta \preceq \Delta \), we define \( Z_{f,\theta} := \overline{Z}_{f,\Delta} \cap T_\theta \), then we have the decomposition of \( Z_{f,\Delta} \) into disjoint unions:

\[
\overline{Z}_{f,\Delta} = \bigcup_{\theta \preceq \Delta} Z_{f,\theta}
\]

**Definition 7.2.** A Laurent polynomial \( f \) with a full dimensional Newton polyhedra \( \Delta \) and the corresponding hypersurface \( Z_{f,\Delta} \subseteq T_\Delta \), \( \overline{Z}_{f,\Delta} \subseteq \mathbb{P}_\Delta \) are said to be \( \Delta \)-regular if for every face \( \Theta \preceq \Delta \) the affine variety \( Z_{f,\Theta} \) is empty or a smooth subvariety of codimension 1 in \( T_\Theta \). We denote the collection of all \( \Delta \)–regular hypersurfaces in \( X_\Delta \) as \( L(\Delta) \). Affine varieties \( Z_{f,\Theta} \) are called the strata on \( \overline{Z}_{f,\Delta} \) associated with faces \( \Theta \subseteq \Delta \).

**Definition 7.3.** Let \( f \) be a Laurent polynomial, \( Z_f \) be the affine hypersurface defined by the Laurent polynomial \( f \) and let \( \overline{Z}_{f,\Sigma} \) be the closure in \( \mathbb{P}_\Sigma \). By orbit cone correspondence theorem, every \( \sigma \in \Sigma \) corresponds to one and only one \( T \)-invariant orbit \( T_\sigma \), thus we have the disjoint union

\[
\overline{Z}_{f,\Sigma} = \bigcup Z_{f,\sigma}
\]

where \( Z_{f,\sigma} = Z_f \cap T_\sigma \). A Laurent polynomial \( f \) and the corresponding hypersurfaces \( Z_f \subseteq T \), \( \overline{Z}_{f,\Sigma} \subseteq \mathbb{P}_\Sigma \) are said to be \( \Sigma \)-regular if for every cone \( \sigma \in \Sigma \), \( Z_{f,\sigma} \) is either empty or a codimension 1 smooth subvariety in \( T_\sigma \).
Definition 7.4. Let \( \varphi \) be an isomorphism between lattices \( N' \) and \( N \). Let \( \Sigma', \Sigma \) be two fans in \( N', N \) respectively. We say \( \varphi(\Sigma') \) is a subdivision of \( \Sigma \) if

1. every cone \( \sigma \in \Sigma \) is a union of cones of \( \varphi(\Sigma') \).
2. for every cone \( \sigma' \in \Sigma' \), \( \varphi(\sigma') \subset \sigma \) for some \( \sigma \Sigma \).

Remark 7.5. As was pointed out in [4], the toric morphism induced by the the aforemen- tioned lattice isomorphism \( \varphi \) gives rise to a proper birational morphism between \( P_{\Sigma'} \) and \( P_\Sigma \). Since the toric morphism is an isomorphism on the tori, \( P_{\Sigma'} \) and \( P_\Sigma \) have isomorphic open subsets. Whence \( P_{\Sigma'} \) and \( P_\Sigma \) are birational.

Proposition 7.6. Let \( \phi : N' \rightarrow N \) be an isomorphism of \( n \)-dimensional lattices, \( \Sigma \) a fan in \( N_Q \), \( \Sigma' \) a fan in \( N'_Q \). Assume that \( \phi(\Sigma') \) is a subdivision of \( \Sigma \). Let

\[
\tilde{\phi} : P_{\Sigma', N'} \rightarrow P_{\Sigma, N}
\]

be the corresponding proper birational morphism of toric varieties. Then for any \( \Sigma \)-regular hypersurface \( Z_\phi \subseteq \Sigma \), the hypersurface \( \tilde{Z}_{\phi}(f) \subseteq P_{\Sigma'} \) is \( \Sigma' \)-regular.

Proof. The proof uses this fact:

\[
Z_{\tilde{\phi}^*f, \sigma} \cong Z_{f, \sigma} \times (\mathbb{C}^*)^{\dim \sigma - \dim \sigma'}
\]

for any \( \sigma' \in \Sigma', \sigma \in \Sigma \) with \( \phi(\sigma') \subseteq \sigma \). Since \( Z_\phi \) is \( \Sigma \)-regular, it is either empty or a codimension one smooth subvariety of \( T_\sigma \). If \( Z_\phi \) is empty, then so is \( Z_{\tilde{\phi}^*f, \sigma} \). If \( Z_\phi \) is a codimension one smooth subvariety of \( T_\sigma \), then \( \dim Z_{\tilde{\phi}^*f, \sigma} = \dim Z_{f, \sigma} + \dim \sigma - \dim \sigma' = (\dim T_\sigma - 1) + \dim \sigma - \dim \sigma' = n - 1 - \dim \sigma' = \dim T_\sigma - 1 \). Moreover \( Z_{\tilde{\phi}^*f, \sigma} \) is smooth since it is a product of nonsingular varieties.

Remark 7.7. Let \( \phi \) and \( \tilde{\phi} \) be the lattice homomorphism and morphism as above. Then

1. \( \tilde{\phi}^{-1}(Z_f) = Z_{\tilde{\phi}^*f} \).
2. If \( f = \sum c_m x^m \), then \( \tilde{\phi}^*(f) = \sum c_m \tilde{\phi}^*(m) \) where \( \tilde{\phi}^* : M \rightarrow M' \) is the dual map of \( \phi \).

Proof. For the first statement, it is sufficient to prove

\[
\tilde{\phi}^{-1}(Z_f) \cap T_{N'} = Z_{\tilde{\phi}^*(f)}.
\]

If \( a \in \tilde{\phi}^{-1}(Z_f) \cap T_{N'} \), then \( \tilde{\phi}(a) \in Z_f \cap T_N = Z_f \). This implies

\[
f(\tilde{\phi}(a)) = (\tilde{\phi}^*f)(a) = 0
\]

so that \( a \in Z_{\tilde{\phi}^*(f)} \). Conversely, given \( b \in Z_{\tilde{\phi}^*(f)} \subseteq T_{N'} \), we have

\[
(\tilde{\phi}^*f)(b) = f(\phi(b)) = 0
\]

so that \( \phi(b) \in Z_f = Z_f \cap T_N \). Hence \( b \in \tilde{\phi}^{-1}(Z_f) \cap T_{N'} \).

For the second statement, we prove that \( \tilde{\phi}^*(x^m) = x^{\phi^*(m)} \). For all \( m_2 \in M_2 \), \( t \in T_{N'} \), where \( t = \lambda^u(c) \) for some \( u \in N' \), \( c \in \mathbb{C}^* \),

\[
\tilde{\phi}^*(x^m)(t) = x^m(\tilde{\phi}(t)) = \chi^m(\phi(u) \otimes c) = c(m, \phi(u)) = c^{\phi(u)(m)} = c^{{\phi^*(m)}(u)} = c^{{\phi^*(m), u}} = \chi^{{\phi^*(m)}(\lambda^u(c))} = \chi^{{\phi^*(m)}(t)}.
\]

Therefore, \( \tilde{\phi}^*(x^m) = x^{\phi^*(m)} \). \qed
Corollary 7.8. Let $\phi : (M_1, \Delta_1) \rightarrow (M_2, \Delta_2)$ be a finite morphism of reflexive pairs. Let $\varphi : P_{\Delta_2} \rightarrow P_{\Delta_1}$ be the corresponding toric morphism. Then for any $\Delta_1$-regular hypersurface $Z_{f, \Delta_1} \subseteq P_{\Delta_1}$, the hypersurface $Z_{\varphi^*(f), \Delta_2} = \varphi^{-1}(Z_{f, \Delta_1})$ is $\Delta_2$-regular.

Proof. Identifying $X_{\Delta_1}$ with $P_{\Sigma_{\Delta_1}}$, and $Z_{f, \Delta_1}$ with $Z_{f, \Sigma_{\Delta_1}}$ we have

$$f, Z_{f, \Delta_1}, \text{ and } Z_{f, \Sigma_{\Delta_1}} \text{ are } \Delta_1\text{-regular} \iff \forall \Theta < \Delta_1, Z_{f, \Delta_1} \cap T_{\Theta} \text{ is either empty or a smooth codimension one subvariety of } T_{\Theta} \iff \forall \sigma \in \Sigma_{\Delta_1}, Z_{f, \Sigma_{\Delta_1}} \text{ is either empty or a smooth codimension one subvariety of } T_{\sigma} \iff \phi^*(f), Z_{\varphi^*(f), \Delta_2}, \text{ and } Z_{\varphi^*(f), \Sigma_{\Delta_2}} \text{ are } \Sigma_{\Delta_2}\text{-regular}.$$ 

Likewise,

$$Z_{\varphi^*(f), \Delta_2} \cap T_{\Theta} \text{ is either empty or a smooth codimension one subvariety of } T_{\Theta} \iff \forall \sigma \in \Sigma_{\Delta_2}, Z_{\varphi^*(f), \Sigma_{\Delta_2}} \text{ is either empty or a smooth codimension one subvariety of } T_{\sigma} \iff \phi^*(f), Z_{\varphi^*(f), \Delta_2}, \text{ and } Z_{\varphi^*(f), \Sigma_{\Delta_2}} \text{ are } \Sigma_{\Delta_2}\text{-regular}.$$ 

Then since $f, Z_{f, \Delta_1}$ and $Z_{f, \Sigma_{\Delta_1}}$ are $\Delta_1$-regular, $f, Z_{f, \Sigma_{\Delta_1}}$ and $Z_{f, \Sigma_{\Delta_1}}$ are $\Sigma_{\Delta_1}$-regular. By reflexivity of $\Delta_1$, $\phi^*(\Sigma_{\Delta_1})$ is a subdivision of $\Sigma_{\Delta_1}$. Therefore, the previous proposition implies $\phi^*(f), Z_{\varphi^*(f), \Delta_2}$, and $Z_{\varphi^*(f), \Sigma_{\Delta_2}}$ are $\Sigma_{\Delta_2}$-regular. Whence $\phi^*(f), Z_{\varphi^*(f), \Delta_2}$, and $Z_{\varphi^*(f), \Sigma_{\Delta_2}}$ are $\Delta_2$-regular.

We return back to our last subject where a finite group $G$ acts on a toric variety.

Proposition 7.9. Let $\phi : (\Delta_1, M_1) \rightarrow (\Delta_2, M_2)$ be a finite morphism of reflexive pairs, $\varphi : X_{\Delta_2} \rightarrow X_{\Delta_1}$ the corresponding toric morphism. Then the hypersurfaces in $L(\Delta_1)$ are quotients of some hypersurfaces in $L(\Delta_2)$.

Proof. We have already seen that $G = N_1, \phi^*(N_2)$ acts on $X_{\Delta_2}$. We claim that $G$ acts on any $\varphi^{-1}(Z_{f, \Delta_1})$ where the action is given by the original action of $G$ on $X_{\Delta_2}$. Let $g \in G$, $x \in \varphi^{-1}(Z_{f, \Delta_1})$. Since $\varphi$ constant on $G$-orbits, we have $\varphi(g \cdot x) = \varphi(x)$. Therefore, $g \cdot x \in \varphi^{-1}(Z_{f, \Delta_1})$. It follows that the $G$-orbits on $\varphi^{-1}(Z_{f, \Delta_1})$ are $T$ where $T$ is a $G$-orbit in $X_{\Delta_2}$ having nonempty intersections with $\varphi^{-1}(Z_{f, \Delta_1})$. Moreover, if $\varphi(x) = \varphi(y)$, then $x$ and $y$ lie in the same orbit in $X_{\Delta_2}$, which is also the orbit in $\varphi^{-1}(Z_{f, \Delta_1})$. Finally, since $\varphi$ is surjective, we have $\varphi(\varphi^{-1}(Z_{f, \Delta_1})) = Z_{f, \Delta_1}$. Whence there is a one to one correspondence between points of $Z_{f, \Delta_1}$ and $G$-orbits in $\varphi^{-1}(Z_{f, \Delta_1})$.}

Corollary 7.10. Let $\phi : (\Delta_1, M_1) \rightarrow (\Delta_2, M_2)$ be a finite morphism of reflexive pairs. If the family of hypersurfaces $L(\Delta_1)$ in $X_{\Delta_2}$ are the quotient of some hypersurfaces of $L(\Delta_2)$ in $X_{\Delta_2}$ by a finite abelian group $A$, then the family $L(\Delta_1^*)$ of hypersurfaces in $X_{\Delta_2}$ are quotients of some hypersurfaces of $L(\Delta_1^*)$ in $X_{\Delta_1}$ by the finite abelian group $A^* = \text{Hom}_{\mathbb{Z}}(A, \mathbb{C}^*)$ dual to $A$.

Proof. Simply observe that $\phi^* : (N_1, \Delta_1^*) \rightarrow (N_2, \Delta_2^*)$ is a finite morphism of reflexive pairs. So everything we have done carries verbatim to the new arguments with only $N_i$ being replaced by $M_i$, $\phi$ being replaced by $\phi^*$ and $\phi^*$ being replaced by $\phi$. 

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We end this paper by refining the relations in the above corollary into Galois correspondence.

**Theorem 7.11.** Let $S = \{ L(\Delta) \mid \Delta \text{ is a reflexive polytope} \}$. Define a relation on $S$ by $L(\Delta_1) \leq L(\Delta_2)$ if there is a finite morphism of reflexive pairs

$$(M_1, \Delta_1) \xrightarrow{\phi} (M_2, \Delta_2).$$

Namely, hypersurfaces in $L(\Delta_1)$ is a quotient of hypersurfaces in $L(\Delta_2)$ by the finite abelian group $N_1/\phi^*(N_2)$. Then there is a Galois correspondence $\theta : S \rightarrow S$ given by $\theta(L(\Delta)) = L(\Delta^*)$.

**Proof.** Let us verify the relation $\leq$ is a partial order on $S$. To begin with, we obviously have $L(\Delta) \leq L(\Delta)$. If $L(\Delta_1) \leq L(\Delta_2)$ and $L(\Delta_2) \leq L(\Delta_1)$, then there are finite morphisms of reflexive pairs $\mu, \nu$ such that $\nu\mu(\Delta_1) = \Delta_1$ and $\mu\nu(\Delta_2) = \Delta_2$. Since for $i=1,2$, $\Delta_i \subset (M_i)_\mathbb{R}$ contains 0 in its interior, it contains a basis for its ambient space $(M_i)_\mathbb{R}$. Therefore, $\mu, \nu$ are linear isomorphisms. This implies there is a toric isomorphism $X_{\Delta_2} \simeq X_{\Delta_1}$. Hence, $L(\Delta_1) \simeq L(\Delta_2)$. If $L(\Delta_1) \leq L(\Delta_2)$ and $L(\Delta_2) \leq L(\Delta_3)$, then there are finite morphisms of reflexive pairs

$$(M_1, \Delta_1) \xrightarrow{\mu} (M_2, \Delta_2) \xrightarrow{\nu} (M_3, \Delta_3).$$

Then $\nu \circ \mu$ will be a finite morphism of reflexive pairs $(M_1, \Delta_1)$ and $(M_3, \Delta_3)$. As a result, $L(\Delta_1) \leq L(\Delta_3)$. By previous corollary, we have

1. $L(\Delta_1) \leq L(\Delta_2)$ if and only if $\theta(L(\Delta_2)) \leq \theta(L(\Delta_1))$.
2. $L = \theta(L^*)$.

Hence $\theta$ is a Galois correspondence between the families of hypersurfaces in Batyrev’s mirror construction from reflexive polytopes. We prove further that if we have two finite morphisms of reflexive pairs

$$(M_1, \Delta_1) \xrightarrow{\mu} (M_2, \Delta_2) \xrightarrow{\nu} (M_3, \Delta_3),$$

then

$$|M_3/\nu\mu(M_1)| = |M_3/\nu(M_2)||M_2/\mu(M_1)|.$$

By third isomorphism theorem, we have the relation

$$\frac{M_3/\nu\mu(M_1)}{\nu(M_2)/\nu\mu(M_1)} \simeq M_3/\nu(M_2).$$

Therefore,

$$|M_3/\nu\mu(M_1)| = |\nu(M_2)/\nu\mu(M_1)||M_3/\nu(M_2)|.$$

However, the assignment $[m_3] \mapsto [\nu(m_2)]^*$ is a well defined bijection between $M_2/\mu(M_1)$ and $\nu(M_2)/\nu\mu(M_1)$. $[\ ]^*$ here stands for the equivalence class in $M_2/\mu(M_1)$ and $[\ ]^*$ in $\nu(M_2)/\nu\mu(M_1)$. So as in field’s version of Galois theory, if we are given a finite morphism
of reflexive pairs \( \phi : (M_1, \Delta_1) \rightarrow (M_3, \Delta_3) \). Then for any finite morphisms \( \nu, \mu \) of reflexive pairs such that the diagram

\[
\begin{array}{ccc}
(M_1, \Delta_1) & \xrightarrow{\phi} & (M_3, \Delta_3) \\
\downarrow{\nu} & & \downarrow{\mu} \\
(M_2, \Delta_2) & & \\
\end{array}
\]

commutes, we have \([M_3 : M_1] = [M_3 : M_2][M_2 : M_1]\) if we define \([M_i : M_j]\) by \([M_i/\text{image of } M_j]\).

\[\square\]

References

[1] Robin Hartshorne, Algebraic Geometry, Springer